UČITELJSKI FAKULTET, SVEUČILIŠTE U ZAGREBU
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Međunarodna znanstvena i umjetnička konferencija
***Suvremene teme u odgoju i obrazovanju – STOO 2019.***

# Zbornik radova simpozija
# ***Novi izazovi u nastavi matematike***

15. – 17. studenog 2019.
Zagreb, Hrvatska

International Scientific and Art Conference
***Contemporary Themes in Education – CTE 2019***

# Proceedings of the Symposium
# ***New Challenges in Mathematics Teaching***

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**Dubravka Glasnović Gracin**, **Goran Trupčević**

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# Simpozij: **Novi izazovi u nastavi matematike**

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# Content / Sadržaj

# Pozvano predavanje / Keynote lecture

# Thinking about – Making Sense
# Reflecting in mathematics lessons – why, about what, how

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## Abstract

Reflecting is seen as an essential activity to be developed (also) in mathematics classrooms. There is a broad consensus in literature referring to mathematics education. Relevant regulations of educational policy such as syllabus, (national) educational standards, international tests like PISA also contain reflecting as required activity. In real mathematics lessons, on the other hand, there is little evidence of reflecting. So there seems to be a considerable problem of matching between claim and reality. This problem of fitting is a key issue in the project "Reflecting in mathematics classrooms" at the Institute of Mathematics Education at the University of Klagenfurt. Interventions are developed and proposed for mathematics lessons.
First, the paper will give insight into the role and meaning attributed to reflecting by mathematics education. In the second part, conceptual definitions are presented which were made in the context of the project. On the one hand, it is a question of what should be meant by reflection in the project and, on the other hand, it is about suggestions, which types of reflection should be integrated into the teaching of mathematics. Thus, it is about the question of what should be reflected on and the question of which objects or relationships are at the center of reflections. These can be inner-mathematical relationships as well as relationships between mathematics and our world or even relationships between mathematics and us humans. The focus of the third part is on concrete tasks that are intended to stimulate reflections in mathematics lessons. As part of the project, a series of tasks addressing all types of reflections were developed for different mathematical topics and different educational levels (especially grades 5-12).

***Key words***: *conceptual definitions of reflecting; role and meaning of reflecting; secondary mathematics education; tasks stimulating reflections; types of reflection;*

## Reflecting, an essential activity of Maths lessons

Doing Mathematics is not limited to correct calculating or operating. Instead of this, Mathematics educators demand more "thoughtfulness“ in mathematics lessons, which means more reflection in mathematics lessons. This thoughtfulness includes manifold reflections with regard to the meaning of mathematical concepts, of mathematical representations, procedures, methods, correlations, relationships, as well as to the meaning and relevance of mathematics as a whole. (e.g. Peschek, W., Prediger, S., & Schneider, E., 2008)

In further text, a few such concepts will be discussed in more detail, with the emphasis being placed on an educational perspective.

R. Fischer (2001, 2012) sees in his concept of *subject-oriented higher general education* the *ability to communicate with experts and the general public* and, directly linked to this, the ability to make decisions as core competences of our society. This demand is based on the observation that the functioning of our society is essentially based on an emancipated and appropriate handling of highly specialized expert knowledge. As responsible citizens, we are confronted with expert statements in many questions of public as well as private life and must form an opinion in order to be able to make decisions. And since we can only be experts in a few areas, we need to be able to communicate with experts in those areas where we are laymen. So, *"it is not a question of becoming an expert in a certain field, but one should be able to communicate with experts" (Fischer, 2001, p. 152, [translated by the author]).* This means asking the experts the right questions, classifying and evaluating their answers and drawing conclusions from them, i.e., making assessments and making decisions.

With regard to such communication skills, Fischer sees two areas of competence as particularly important: *basic knowledge and reflection*. Basic knowledge is a prerequisite for communication with experts and reflection is seen as necessary:

- to classify statements made by experts into the own problem area,
- to be able to evaluate them and
- to come to well-considered decisions.

For mathematics teaching, the following can be taken from Fischer's concept regarding the relevance of reflections:

- Reflection is important as support for the development of basic knowledge and deeper understanding of fundamental mathematical terms, representations, concepts.
- Reflection is important to build up relevant reflection knowledge. We expect that the product that is developed in the process of reflecting will become significant knowledge. We call this knowledge reflection knowledge.
- Fischer himself focuses on reflecting on the meaning and significance of a mathematical content for oneself, for certain communities, and for society. Fischer is concerned here with an explicit discussion of the meaning and significance of a mathematical content (concepts, representations, …) by the learners themselves. In this process of dealing with "*the question: 'What do the contents mean to me, what do they mean to society, what do they mean to us as a life community ...', education takes place.*" (Fischer, 2001, p. 158, [translated by the author])

K. Lengnink (2005) proposes that mathematics teaching should enable students to deal with mathematics in a responsible and mature way; she calls it mathematical responsibility and maturity ("*mathematische Mündigkeit*"). The characteristic of mathematical "Mündigkeit" is a critical relationship of the person to mathematics, his/her ability and attitude of self-determination and free determination in all social mathematical decisions. Lengnink sees the development of "Mündigkeit" as a process of discursive consideration of the relationship between human being and mathematics. In this process, reflecting on mathematics and judging mathematics are regarded as important activities and thus important for the teaching of mathematics and for the curriculum. Four aspects of reflection are relevant according to Lengnink: (i) reflection on the mathematical content of a situation, which means thinking about what can or cannot be grasped mathematically about the situation; (ii) reflection on the sense and meaning of basic mathematical terms and concepts; (iii) reflection on the relationship between mathematics and the world, specifically between mathematical models and context-oriented situations, and (iv) reflection on personal attitudes towards mathematics and its applications.

Thus, the focus of reflection with referring to mathematical "Mündigkeit" lies on the relationships between a human being and mathematics and between mathematics and the world.

O. Skovsmose's (1992, 1998) approach to *Critical Mathematics Education* is based on the "*formatting power of mathematics*" (Skovsmose, 1998, p 197) and its social and political significance for a democratic society. "*Social phenomena are structured and eventually constituted by mathematics*" often as a hidden and invisible part (Skovsmose, 1998, p 197). This requires strong analytical tools to recognize the role of mathematics. It is an essential task of mathematics education to impart such tools and thus to promote the development of a critical capacity towards mathematics and its use and effect in social contexts (critical mathematics education). Mathematics teaching should produce "*critical readers of the formatting*" (Skovsmose, 1998, p 197). Skovsmose sees reflecting as an essential activity, whereby the focus of reflection should be on:

- mathematical concepts and algorithms,
- the relationship between mathematics and extra-mathematical reality,
- the social and political function of applying mathematics to a certain situation,
- the importance of mathematics to handle lifeworld problems (e.g. Skovsmose, 1998, pp. 199/200).

In contrast to Lengnink or Fischer, Skovsmose is not concerned with the subject and its relationship to mathematics, but with the effect of mathematics on the world.

Reflection is also mentioned as a requirement in current *educational policy guidelines and instruments*.

In the starting *framework of PISA (*Programme for International Student Assessment), reflection is one of three competency clusters besides reproduction and connections. In this context, reflection refers to thinking about the *mathematical processes or activities* that are needed to solve the given problem (OECD, 2009). The definition of reflection made in the framework of PISA thus differs from the previously discussed ones. Reflection here essentially focuses on problem solving and not on mathematical concepts and their relationship to the world or to the subject.

In the concept of the *Austrian educational standards for mathematics* (for the 8th grade) developed at the Institute of Mathematics Education at the University of Klagenfurt, mathematical competences are characterized as a three-dimensional construct. One of these dimensions is the complexity dimension, which focuses on the kind and complexity of cognitive activities which are required for solving the given problem. The complexity dimension includes three complexity areas and one of them is "reflecting, use of reflection knowledge ". Here, the focus of reflection is on the mathematical concepts and on mathematical models (and not on the mathematical processes). (e.g. Peschek, 2012)

Also, in the *Austrian curriculum for Mathematics,* both for secondary level I (grade 5-8) and secondary level II (grade 9 - 12), reflection is explicitly mentioned (bm:bwf).

In *summary*, reflection is seen by mathematics didactics as an essential mathematical activity. Corresponding activities, skills and attitudes are fundamental for mathematical education and they are to be developed in mathematics teaching in an appropriate way. Reflection is explicitly mentioned as a requirement in educational policy guidelines and instruments relevant to the teaching of mathematics.

However, reflection seems to have hardly caught up with mathematics lessons. This can be deduced from a lot of experiences in long-term teacher training programmes, from discussions with mathematics teachers at the partner schools of our institute, as well as from the lack of knowledge and skills of our students. Textbooks also hardly help teachers to integrate reflection tasks in mathematics lessons - at least in Austria. In Austrian textbooks there are hardly any reflection tasks. A comprehensive analysis of numerous Austrian mathematics textbooks for the 5th to 12th grade, including a total of 12,000 tasks, has shown that the proportion of reflection tasks among them is, on average, between 0 and 1.2 percent per textbook (e.g. Četić, 2018; Deweis, 2018).

There is thus an obvious discrepancy between the claim of mathematics education and the reality in classroom.

## What is meant by reflection?

In the literature, reflection is understood to mean a wide variety of things whereby the focus of reflection is sometimes on different aspects. In our project we have defined the following characterisations (see also Schneider, 2018):

*Reflection* (related to the learning of mathematics in school) means thinking about characteristics, connections, relationships, effects or meanings that cannot directly be read from the given fact.

Following Fischer (2001, 2012), Peschek (2005), Skovsmose (1992, 1998) and Lengnink (2005), four types of reflection are distinguished, each of which was concretized by a variety of questions in the course of the definition of the terms used in the project. In the text below, some questions are given as examples for each type of reflection.

*Mathematics-oriented* reflection: thinking about mathematical properties of mathematical concepts (mathematical objects, representations, procedures, theorems, etc.) and about mathematical relationships within or between such concepts.

The focus of reflection is on mathematics itself, on inner-mathematical properties, connections, and relationships.

Some exemplary questions are:

- Which mathematical objects (do not) have a certain property?
- Which strengths/weaknesses does a certain representation of a mathematical situation have?
- What are the connections/relationships between certain mathematical concepts?
- Does a certain mathematical rule apply or not? Why?
- What are the global ideas of a particular mathematical topic? What is characteristic for a particular mathematical topic?

*Model-oriented reflection*: Thinking about relations between mathematical concepts and inner-mathematical, but above all extra-mathematical situations.

The focus of reflection is on the relationship between mathematics and the world, on thinking about mathematical models, usually of extra-mathematical situations and their fit, limits, effects, and implicit assumptions for the concrete inner-mathematical or extra-mathematical situation.

Some exemplary questions are:

- Why is the mathematical model (not) an appropriate description for the present (extra-mathematical) situation?
- How does modelling with different models affect model results?
- Which implicit assumptions have been used in a particular modelling?
- What are the limits of a particular modelling?

*Context-oriented reflection*: Thinking about the effects of mathematical concepts in our world.

Reflection focuses on mathematizations in our world and on thinking about their (often hidden) social function/effect.

Some exemplary questions are:

- For what purpose is a particular mathematical concept used in the present context (or in other societal contexts)?
- What is the function of mathematization, who does it serve, and what is its purpose?
- What effect does a certain mathematization have? What are the advantages and disadvantages of this? What if we did not have this mathematical concept disposal?
- How does the use of a certain mathematical concept influence our ideas of the context, of social contexts, of "reality", of our (living) world?

*Subject-oriented reflection:* Thinking about the importance and relevance of knowing mathematical concepts and topics for oneself or for certain communities or for society.

The focus of reflection is on the assessment of the significance and benefits of certain mathematical concepts and contents, on the one hand for oneself and on the other hand for certain communities or for society. Mathematical content can mean a very concrete content (e.g. arithmetic mean), but also a whole content area (e.g. descriptive statistics).

Some exemplary questions:

- What does a certain mathematical content mean *to me* personally or what does it mean *to me* as part of a community (e.g. family, class, etc.) or as a member of our society?
- What are the benefits of being familiar with this content for me? Where can or must I use mathematical knowledge and skills now or in later life?
- What benefit does familiarity with a certain mathematical content have *for certain communities* (e.g. class, school, family, friends, etc.), *for our society*? What problems or difficulties could arise if this is not the case?

Subject-oriented reflection is about the assessments of mathematical contents, about individual opinions and perspectives. Both positive and negative assessments are welcome. It is important that the individual positions are explained and argued.

## Some reflection tasks

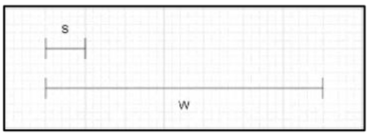
For the types of reflection outlined above, the project has developed a series of tasks stimulating reflections for relevant content areas of secondary levels I and II. Some of these tasks are presented below.

*Mathematics-oriented reflection*

**Different representations**

*verbal*: Diluted raspberry juice contains seven times as much water as syrup.

*graphical*:




*symbolic*: $S \cdot 7 = W$

Which representation has which strength(s)?

*Figure 1. Task for mathematics-oriented reflection (e.g. Deweis-Weidlinger, 2019a)*

The opportunity for reflection in Figure 1, an example of mathematics-oriented reflection, focuses on dealing with three frequently occurring mathematical forms of representation of a situation. The students should think about which aspects are particularly well expressed in each representation. Depending on students' previous knowledge and experience, considerations that are strongly oriented to the context or considerations of a more general nature will be possible. Since reflection tasks are characterized by the fact that there is not a single correct answer, forms of teaching that enable an exchange with other students are almost obligatory.

**Local and global extrema**
Can a local maximum in a closed interval also be a global maximum and vice versa? Explain your answer.
Can a function have multiple local or global minima in a closed interval? Explain your answer.
Can a local maximum in a closed interval also be a global minimum? Explain your answer!

*Figure 2. Task for mathematics-oriented reflection (e.g. Cetic, 2019a)*

The task presented in Figure 2 aims to deal more closely with the relationship between local and global extrema. The focus is on reflecting on the similarities and differences of the two concepts and on possible graphical representations. Students should not be satisfied with just one example but they should consider different cases.

*Model-oriented reflection*

**Average salaries**
The following table shows the arithmetic mean and the median of the monthly gross salaries of employees (total and separate, for women and men) in 2017:

| *Monthly gross salaries of the employees in 2017 (in €)* | | | |
|---|---|---|---|
| | *total* | *women* | *men* |
| *arithmetic mean* | 2,780.00 | 1,950.00 | 3,770.00 |
| *median* | 2,250.00 | 1,690.00 | 3,250.00 |

Why does the average of monthly gross salaries of employees give significantly different values when modelled by the arithmetic mean and by the median?

*Figure 3. Task for model-oriented reflection (e.g. Schneider, 2019)*

The intention of the task in Figure 3 is reflecting on the effect of two different mathematical models in a concrete situation – in one case the arithmetic mean, in the other case the median. Here, we have an obvious asymmetric distribution of the salaries. The two models deal with this phenomenon in different ways.

**Development of the population**
Under what assumption could the development of a population be described in a meaningful linear way? Why does (not) such an assumption usually seem to be realistic?

*Figure 4. Task for model-oriented reflection (e.g. Cetic, 2019b)*

The focus of the example of model-oriented reflection in Figure 4 is on thinking about assumptions used in modelling a special situation, in the specific situation, illustrated by a linear function. It is not a question of reproducing the definition of a linear function but of considering the conditions under which variables relevant to population development such as the number of births, deaths, immigrants, emigrants, etc. must interact in the case of linear modelling. This leads to the second question which deals with whether such assumptions and thus such modelling could be realistic at all.

*Context-oriented reflection*

**Formulas - why?**
Formulas are of great importance both within mathematics and in its applications. In mathematics lessons we have learned about formulas in many different contexts.
For what purpose are formulas used? Explain using examples.

*Figure 5. Task for context-oriented reflection (e.g. Deweis-Weidlinger, 2019a)*

Formulas are something typical and characteristic of mathematics. So, it is obvious to think about what function such formulas could have (in the world). In the task presented in Figure 5, what is in the foreground is the process of reflecting on the purpose of mathematization, not the product of reflection.

**A world without coordinate systems**
What would not be possible if no coordinate systems were available?

*Figure 6. Task for context-oriented reflection (e.g. Deweis-Weidlinger, 2019b)*

**Why different numeral systems?**
A) Why do we need natural numbers?
B) Why do we need negative numbers?
C) Why do we need fractions?
D) Why do we need irrational numbers?

*Figure 7. Task for context-oriented reflection (e.g. Deweis-Weidlinger, 2019c)*

The tasks in Figure 6 and Figure 7 have the same focus: Thinking about the function of mathematical concepts and about what one would lose if these mathematical concepts were not available. In Figure 6 the emphasis of reflection is on the coordinate systems, while in Figure 7 the emphasis is on numeral systems.

*Subject-oriented reflection*

**Plea for/against descriptive statistics**
Imagine that the Ministry of Education is planning to remove descriptive statistics from curricula.
Write a letter to the editor of a daily newspaper in which you explain and justify your opinion of the Ministry of Education's plans.

*Figure 8. Task for subject-oriented reflection (e.g. Schneider, 2019)*

**Pros and cons of stochastics**
In a well-known Austrian daily newspaper, under the heading "Comment", there is sometimes one comment "pro" and one "contra" of a current statement.
Write one pro and one contra (each max. 200 words) of the statement:
"*Probability and Inferential statistics should be deleted from the Mathematics curriculum.*"

*Figure 9. Task for subject-oriented reflection (e.g. Schneider, 2019)*

Both reflection tasks aim to reflect on the importance and relevance of a mathematical topic, descriptive statistics (Figure 8) and stochastics (Figure 9) for oneself, but particularly for the society. Removing descriptive statistics or stochastics from the curriculum would concern all. In the first example (Figure 8) it is free to the students whether they will take a pro or a contra point of view. However, it is important that their point of view is explained and justified. In the second example (Figure 9) arguments are deliberately demanded for and against. A differentiated argumentation is desired.

**Algebra - Is it important?**
We have been dealing with algebra for a long time and have learned a lot about it.

A Is Algebra important for you personally, today or for your later life? Why (not)? Try to explain this with concrete examples/situations.

B Should all students learn algebra and why (not)? Try to explain this using concrete examples/situations.

*Figure 10. Task for subject-oriented reflection (e.g. Deweis-Weidlinger, 2019a)*

The focus of the task presented in Figure 10 is also on reflecting on the significance of a mathematical topic. However, the task is divided into two parts. In part A, the students should reflect on the benefit of the specific mathematical content for themselves. In part B, they should think about whether all students should learn this mathematical content. Different answers for parts A and B are possible.

## Review and outlook

It is not about the question of eliminating the important operational activities from mathematics teaching, but of enriching the Mathematics lessons through the equally essential mathematical activity of reflecting.

Without doubt, it can also be important (and necessary) to transmit reflection knowledge to the students. But it is also very important that students reflect independently. The activity of reflection should be considered as an important and essential mathematical activity. This activity of reflecting should also become a basic human attitude – also outside mathematics.

So far, we have noticed deficits in reflection in mathematics lessons (e.g. textbook analysis). In the project, we have confirmed the relevance of reflection for mathematical education and developed a concept of reflection based on didactic literature that can be used constructively and analytically.

An *empirical study* is carried out in the current school year, 2019/20. The aim of this study is to gain insights into how successful is reflection-oriented mathematics teaching ? How do students deal with reflection tasks? How do teachers deal with reflection tasks? Can changes be observed with continuous use of reflection tasks in the classroom? What ideas or attitudes are there regarding reflection and reflection tasks, provided both by the students and by the teachers? Can changes be observed in the course of the school year?

Six teachers from three different grammar schools, each with one class, have participated in the study. The spectrum of classes spans from the 6th to the 11th grade. Reflection tasks are used by the teachers throughout the school year on various mathematical topics. The use of the reflection tasks is accompanied by the project team meetings (regular meetings with the teachers) and evaluation (class observations, observation protocols, interviews).

The results of this study will be six case studies on the use of reflection tasks in teaching throughout a school year. In this way, the project not only provides a concept for reflection in mathematics teaching but also a compilation of reflection tasks tested in the classroom on all main topics of mathematics teaching. The aim is also to gain further experiences, insights and recommendations for the teaching of reflection tasks which will subsequently be made available to teachers in collections of materials, publications or in teacher training events.

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# Cjeloviti radovi / Full papers

# Utilizing the task progressions framework to support lesson design

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## Abstract

The author describes interactions with teachers as they utilize the Task Progressions Framework (Courtney & Glasnović Gracin, 2019) to develop a sequence of mathematics lessons. As described in Courtney and Glasnović Gracin (2019), the Task Progressions Framework integrates components of existing task analysis frameworks, notions of 'rich' tasks, differentiated instruction, task format, and learning progressions to develop a guide for use in assembling (for teachers) or analysing (for researchers) a set of tasks that help students develop particular mathematical ideas and particular mathematical habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996). Whereas Courtney and Glasnović Gracin (2019) focused on the framework's development and utilization in analysing the quality, diversity, and complexity of tasks in a lesson or sequence of lessons, the focus of this paper is on the research question: How does the Task Progressions Framework help teachers develop mathematics lessons comprised of a focused, coherent, inclusive, and rich sequence of tasks? The author worked with two 8th grade (student ages 13-14 years) and three Algebra 1 (student ages 14-16 years) mathematics teachers in the midwestern U.S. to design lessons covering content chosen by each group of teachers. Teachers were asked to focus on three of the four dimensions of the Task Progressions Framework (i.e. Content, Mathematical Habits of Mind, Task Format), and allowed to focus their lessons on specific dimension sub-categories, such as: Level of Task Complexity or Rigor and Expectations of Student Products. Teachers were also required to situate their tasks along a learning progression, identified as "key waypoints along the path in which students' knowledge and skills are likely to grow and develop in [mathematics]" (Daro, Mosher, & Corcoran, 2011, p. 12). Results highlight the complex nature of inclusive classroom environments and the multitude of expectations required of and limited supports provided to mathematics teachers.

***Key words**: Lesson planning, mathematical habits of mind, learning progressions, task analysis*

## Introduction

At the *Eleventh Congress of the European Society for Research in Mathematics Education* (CERME11), Courtney and Glasnović Gracin (2019) introduced the *Task Progressions Framework*. The framework integrates components of existing task analysis frameworks (Glasnović Gracin, 2018), notions of 'rich' tasks (Courtney, Caniglia, & Singh, 2014), differentiated instruction, task format, and learning progressions to develop a guide for use in assembling (for teachers) or analysing (for researchers) a set of tasks that help students develop particular mathematical ideas and particular mathematical habits of mind (e.g., Cuoco, Goldenberg, & Mark, 1996). As such, the framework allows teachers and researchers to ascertain the focus, coherence, inclusiveness, and 'richness' of a sequence of tasks comprising a lesson or sequence of lessons. The framework, as presented in Courtney and Glasnović Gracin (2019), is illustrated in Figure 1.

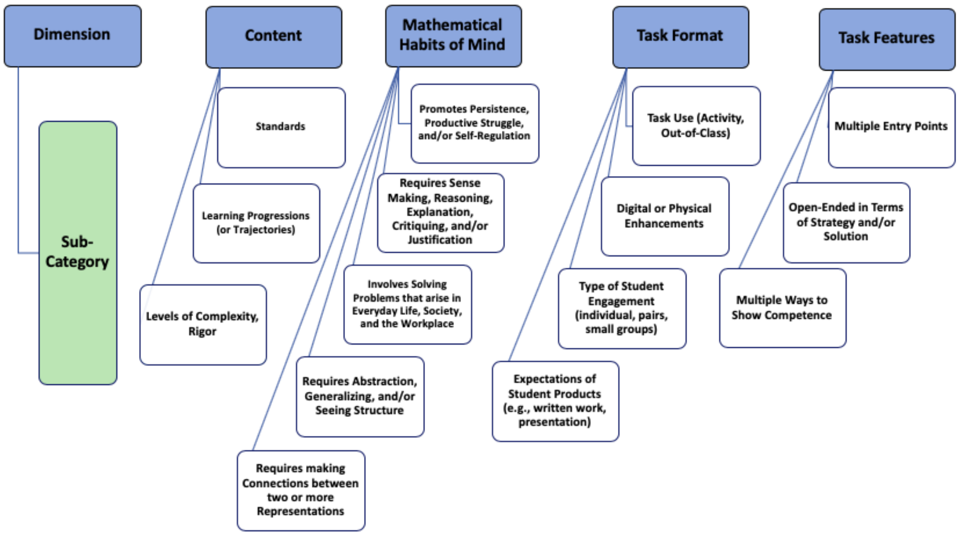


*Figure 1. Task Progressions Framework*

Courtney and Glasnović Gracin (2019) utilized aspects of the framework to analyse a sequence of 8th grade mathematics lessons (covering approximately 80 minutes of instruction) involving the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010). The lessons were created by an 8th grade math teacher to help his students (ages 13-14 years) develop proficiency with the content standard identified by the alphanumeric indicator 8.EE.8a (8: 8th Grade Mathematics, EE: Expressions and Equations domain, 8a: Part a of the eighth standard in this domain) and given as: "Understand that the solution to a pair of linear equations in two variables corresponds to the point(s) of intersection of their graphs, because the point(s) of intersection satisfy both equations simultaneously" (NGA Center & CCSSO, 2010, p. 55). All tasks used during the lessons derived from the class textbook Big Ideas Math: Modeling Real Life, Grade 8 (Larson & Boswell, 2018). In order to analyse the sequence of lessons (i.e. to analyse the quality, diversity, complexity, and coherence of tasks and activities in the sequence of lessons), Courtney and Glasnović Gracin (2019) proposed a scoring system where each dimension or category (e.g. Task Format) is viewed as playing a unique role in the framework, and each sub-category or sub-dimension (e.g. Type of Student Engagement) is of equal importance within a given dimension (i.e. there is no hierarchy within a dimension). Finally, each of the four dimensions or categories were given equal weight (1/4 of total score), regardless of the number of sub-dimensions or sub-categories.

This paper focuses on the alternative role envisioned for the framework, that of its use by in-service (or practicing) teachers as they assemble tasks and activities to develop a sequence of lessons. Here, I present preliminary results situated within a larger project designed to examine the resources grades 6-12 mathematics teachers and math intervention specialists (i.e. special educators) utilize as they plan for instruction. By resources, the project refers to curriculum resources, defined by Pepin and Gueudet (2018) as "all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (p. 132). Such resources include text resources (e.g., textbooks, teacher curricular guidelines, websites, worksheets, syllabi, tests); other material resources (e.g., manipulatives, calculators); digital-/ICT-based curriculum resources (e.g., interactive e-textbooks) (Pepin & Gueudet, 2018, p. 132). Furthermore, resources include "discussions between teachers, orally or online" (Gueudet & Trouche, 2009, p. 200); students' written work; teachers discussions with mathematics teacher educators; and so forth. Teachers interact with resources, select them and work on and with them (e.g., adapting, modifying, reorganizing) within processes where design and enacting are intertwined. Furthermore, teachers are introduced to and interact with resources through discussions with colleagues, professional learning experiences, and via focused or random online searches. The focus of this paper is on the research question: How does the Task Progressions Framework help teachers develop mathematics lessons comprised of a focused, coherent, inclusive, and rich sequence of tasks?

## Methods

The author worked with two middle grades (grade 8; student ages 13-14 years) and three high school (Algebra 1; students ages 14-16 years) mathematics teachers in the midwestern U.S. to design lessons covering content chosen by each teacher. Although teachers were introduced to all four primary dimensions or categories of the Task Progressions Framework, it was requested they focus their lessons on the sub-categories (or sub-dimensions) from only Content, Mathematical Habits of Mind, and Task Format, such as: Level of Task Complexity or Rigor; or Promotes Persistence, Productive Struggle, and/or Self-Regulation (see Figure 1). This restriction was designed to motivate teachers to think about the alignment, coherence, and inclusiveness of content standards, habits of mind, tasks, activities, and assessments across a sequence of lessons. In addition, teachers were required to situate their tasks along a learning progression, identified as "key waypoints along the path in which students' knowledge and skills are likely to grow and develop in [mathematics]" (Daro, Mosher, & Corcoran, 2011, p. 12). Interactions with participating teachers involved five online (using Google Hangout) and three face-to-face conversations and observations as teachers discussed and developed their sequence of lessons. These discussions included collaborative sessions, where participants teaching the same grade level (i.e. Grade 8 Mathematics) or course (Algebra 1) met online to develop lessons using a modified version of "backward design" (Wiggins & McTighe, 2005).

According to Wiggins and McTighe (2005), backward design "involves thinking a great deal, first, about the specific learning sought, and the evidence of such learnings, before thinking about what we, as the teacher, will do or provide in teaching and learning activities" (p. 14). Teachers were asked to develop assessment tasks they believed would faithfully evaluate their students' capacities to engage in and exhibit previously identified mathematical content standards and associated mathematical habits of mind, keeping in mind their diverse student populations. It was requested that these summative assessment (i.e. post-assessment) tasks made clear which specific understandings teachers intended to assess and what such understandings looked like in practice. After creating their summative assessment tasks (i.e. post-assessment), teachers were asked to create a diagnostic (or pre-assessment) to elicit evidence of student learning of pre-requisite knowledge (e.g. understandings, skills, terminology, habits of mind) needed to participate productively in the lessons, along with some mathematics content to be covered in the upcoming lessons. Such diagnostics have the potential to allow for: a) condensing of the lessons, by focusing only on the topics with which students might struggle, and b) differentiated instruction, by providing additional support to some students and giving students who demonstrate understanding of the concepts extensions or challenge material during the lessons (Connor, 2015). Once the post-assessment tasks and diagnostic were created - the "bookends" to the sequence of lessons - teachers were asked to "fill in" the gap by developing a sequence of lessons that would utilize the data generated from the diagnostic (pre-assessment) and prepare their students to be successful on the summative assessment (post-assessment) tasks, see Figure 2.

Collaborative discussions also included focusing teachers on situating their sequence of lessons within a larger "unit". This larger unit is then situated within the curriculum for a specific grade level (e.g. Grade 8 Mathematics) or course (e.g. Algebra 1). This grade level or course, in turn, is situated within the curriculum for their school or school district (e.g. Grades K-8 Mathematics, Algebra 1, Geometry, Algebra 2, Precalculus, Calculus), as illustrated in Figure 2.

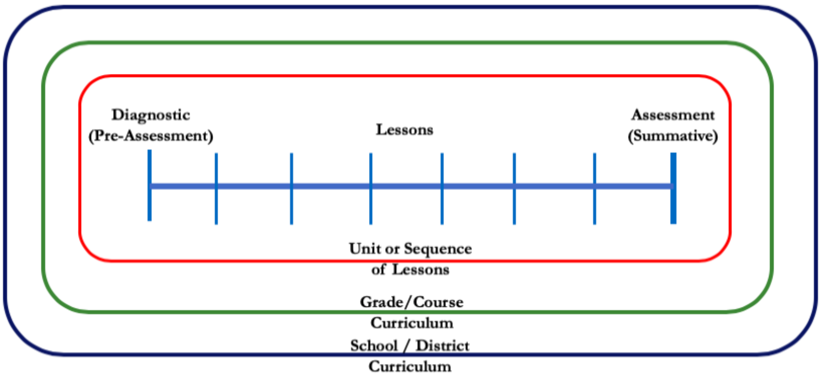


*Figure 2. Lesson Progression*

In the following sections, I characterize participating teachers' attempts to utilize the Task Progressions Framework and describe modifications that were made to the framework to make it more practical for teachers (as they create lessons) and researchers (as they analyse lessons).

## Results

As teachers developed their diagnostic (i.e. pre-assessment), they were introduced to the *Coherence Map* tool (Student Achievement Partners [SAP], n.d.). According to Student Achievement Partners (n.d.) - a non-profit organization in the U.S. dedicated to helping teachers and administrators implement high-quality, college- and career-ready standards - the *Coherence Map* is an interactive digital tool that illustrates connections between the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010) by linking together concepts within and across grade levels or courses. For example, connections for the content standard with alphanumeric indicator 8.EE.5 (8: 8th Grade Mathematics, EE: Expressions and Equations domain, 5: Fifth standard in this domain), and standard statement: "Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways" (NGA Center & CCSSO, 2010. P. 55) are illustrated in Figure 3. In Figure 3, the dashed line indicates content standard 8.EE.6 ("Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b") is "related" to 8.EE.5 and could potentially be taught concurrently. In addition, using the terminology of the *Coherence Map* tool (SAP, n.d.), a student who cannot meet content standard 7.RP.2 (7: 7th Grade Mathematics, RP: Ratio and Proportional Relationships domain, 5: Second standard in this domain) statement: "Recognize and represent proportional relationships between quantities" (NGA Center & CCSSO, 2010, p. 48), is not likely to be able to meet content standard 8.EE.5. Similarly, a student who cannot meet content standards 6.RP.2 ("Understand the concept of a unit rate a/b associated with a ratio a:b with b not equal to 0, and use rate language in the context of a ratio relationship"), 6.RP.3 ("Use ratio and rate reasoning to solve real-world and mathematical problems"), and 7.RP.1 ("Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units") is not likely to be able to meet content standard 7.RP.2.

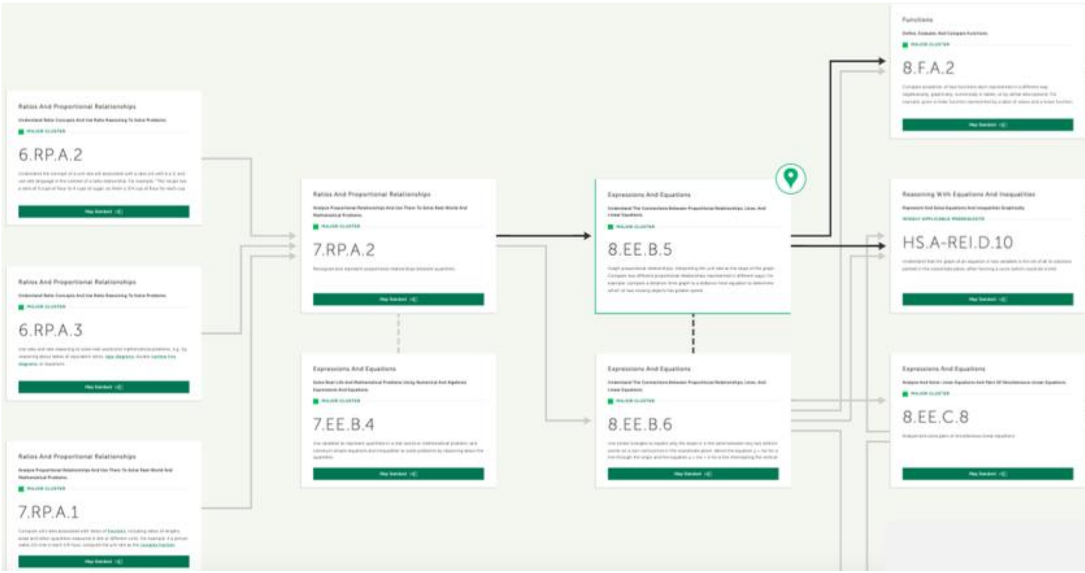


*Figure 3. Coherence Map, Standard 8.EE.5 (Student Achievement Partners, n.d.)*

Although the *Coherence Map* tool was introduced to provoke teachers to reflect on instruction that builds on students' prior learning, participating teachers began to view the Task Progressions Framework as being mainly about the "progression" of content standards. As such, the introduction of the *Coherence Map* tool appeared to constrain teachers as they developed their sequence of lessons using the "bookend" design described earlier. For example, although the two middle grades teachers indicated the *Coherence Map* tool (SAP, n.d.) supported their creation of pre-assessment tasks, the three secondary school teachers asserted the tool identified "too much" knowledge and skills to assess in any practical manner. Therefore, teachers were asked to reflect on their conceptions of "Progression" in the Task Progressions Framework as a progression of tasks and activities - designed to develop or assess students' knowledge and skills - not simply as a progression of content standards.

In addition to the issues involving "progression" described above, teachers also struggled to differentiate lessons and assessments to meet the needs of their diverse student populations. All five participating teachers work in inclusive classroom environments (i.e. general education settings in which students with and without disabilities learn together). Therefore, differentiation was crucial to their development and implementation of tasks, activities, lessons, and assessments. Unfortunately, all five teachers found it challenging to attempt to differentiate instruction and assessments using only the categories Content, Habits of Mind, and Task Format in the Task Progressions Framework. Although interactions with teachers included discussions about using the sub-categories Level of Complexity or Rigor (from the Content category) and Type of Student Engagement and Expectations of Student Products (both from the Task Format category) to support differentiation, teachers indicated a need for more explicit support.

In addition, two participating teachers (one 8th grade, one Algebra 1) described their district's use of the *Universal Design for Learning* (UDL) framework to guide the "design of instructional goals, assessments, methods, and materials . . . to meet individual needs" (CAST, 2018). These two teachers asserted the Task Progressions Framework provided no explicit support using UDL guidelines, such as "Provide multiple means of representation" (CAST, 2018). Furthermore, a second Algebra 1 teacher indicated their district's recent introduction to *Culturally Responsive Teaching* (CRT), which "focuses on elevating the learning *capacity* of students who have traditionally been marginalized in education" (Gunn, 2018, Culturally responsive pedagogy section). This teacher highlighted the framework's lack of explicit support for the various forms of equity-focused education, including social justice education, culturally responsive pedagogy, and trauma-informed education (Gunn, 2018).

## Discussion

As anticipated in Courtney and Glasnović Gracin (2019), results from discussions with and observations of participating teachers highlighted the need to modify portions of the Task Progressions Framework. Specifically, teachers identified the need to support equity-focused education (e.g. Culturally Responsive Teaching) and to make support for differentiation more explicit (e.g. Universal Design for Learning). As a result, the Task Progressions Framework was modified to make equity and the needs of all students more explicit by adding a "Teaching and Learning Environment" category or dimension (see Figure 4).

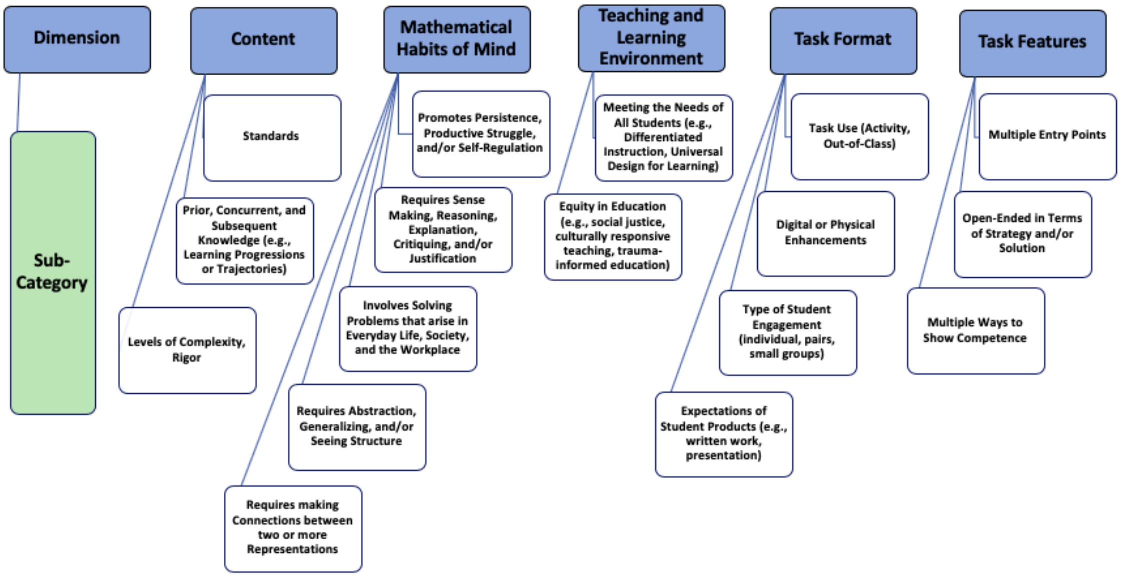


*Figure 4. Task Progressions Framework*

The Teaching and Learning Environment category is comprised of two sub-categories: Meeting the Needs of All Students (e.g. Differentiated Instruction, Universal Design for Learning) and Equity in Education (e.g. Social Justice, Culturally Responsive Teaching, Trauma-Informed Education), as illustrated in Figure 4. The Algebra 1 teachers also struggled to productively employ the *Coherence Map* tool (SAP, n.d.) to focus on the Learning Progressions (or Trajectories) sub-category illustrated in Figure 1. These struggles provoked reflection on whether the issue derived from teachers' use of the *Coherence Map* tool itself or from teachers' conceptions of learning progressions (or trajectories). Furthermore, in order for the Learning Progressions (or Trajectories) sub-category to be meaningful to teachers across countries and cultures, learning progressions (or trajectories) - and potentially tools similar to the *Coherence Map* (SAP, n.d.) - would need to be part of each country's curriculum or standards. Finally, the notion and importance of learning progressions (or trajectories) is not consistent.

According to Lehrer (2013), "[L]earning progressions do not accord well with metaphors of ladders or pathways of development" (p. 173). Rather, aligned with Galison (1997), it "is more profitable to consider learning progressions as a trading zone… in which different realms of educational practice intertwine, much as a cable is constructed" (Lehrer, 2013, p. 183). Similarly, Thompson, Carlson, Byerley, and Hatfield (2014) avoid using the term progression due to its association with "one thing happening after another - a progression in steps" (p. 14) and occur "one idea at a time, in isolation of others" (p. 15). Thompson et al. (2014) seek to convey an image of parallel developments, where understandings and ways of thinking build on prior meanings, are idiosyncratic, and are always in interaction (p. 14). Therefore, for Thompson et al. (2014), the development of understanding and ways of thinking are characterized as the “formation of a learning cloud where many forms of thinking participate in each other’s operation and in each other’s development” (p. 22). As a result of these considerations, the Task Progressions Framework was modified to focus on prior, concurrent, and subsequent knowledge and skills, rather than specific learning progressions or learning trajectories (see Figure 4).

## Conclusion

Results presented here highlight the complex nature of inclusive classrooms. Such environments have become more prevalent in the U.S. and internationally. In the U.S., 63.1% of students with disabilities spend at least 80% of the school day “being educated alongside their typically-developing peers” (Diament, 2019, para. 1). Results also highlight the multitude of expectations required of and limited supports provided to (mathematics) teachers. Furthermore, these results highlight the fact that mathematics teachers do not simply teach mathematics; rather, they teach and learn mathematics to and with humans.

It must be noted that interactions with and observations of teachers utilizing the Task Progressions Framework occurred for only a portion of teachers’ academic year; that is, I have only a snapshot of teachers’ interactions with aspects of the framework for a single sequence of lessons, rather than an entire year’s worth of lessons. Such limitations do not allow for broad and robust analyses of teachers’ interactions with the Task Progressions Framework - something future research must address.

Finally, although I continue to work with in-service (or practicing) 6th-12th grade mathematics teachers (students ages 11-18 years) in the U.S. to utilize the Task Progressions Framework in their practices, attempts must be made to work with teachers in countries other than the U.S., with different affordances and constraints. Such interactions will likely lead to further modifications to the framework to make it more practical for teachers (as they create lessons) and researchers (as they analyse lessons) internationally.

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# "What is geometry for you? Draw a picture."
# Young students' understandings of geometry revealed through drawings

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## Abstract

In the past several decades, the proportion of geometry has been reduced in many national curricula over the world. One of the reasons for this was the temptation to increase the coverage of other mathematical disciplines within school mathematics curriculum. Furthermore, the geometry curriculum has been criticized for its incoherence. These developments raise a question concerning geometry concepts, structures and ideas that students acquire in mathematics education. This question is particularly important in early grades for two reasons. Firstly, this is an important period for the development of geometric thinking. Secondly, it builds a basis for later geometry acquisition. The study presented in this paper focuses on the analysis of elementary students' understanding of geometry by using drawings and a semi-structured interview. The participants were 249 Croatian elementary students (grades 2 to 4). The students were given a piece of paper with the assignment to draw what geometry for him/her is. The analysis of students' data was conducted with respect to the adapted Wittmann's model of seven fundamental ideas of geometry as a theoretical perspective. The results revealed that elementary students have a rather narrow understanding of geometry with respect to exhibited fundamental ideas. Specifically, the fundamental idea of "Geometric forms and their construction" dominated in the students' drawings regardless of the grade level, whereas other fundamental ideas were minimally present, if at all. These results raise issues regarding re-questioning the primary mathematics curriculum requirements concerning the multi-dimensional nature of geometry.

***Key words**: drawings; fundamental ideas; geometry education; primary grade education*

## Introduction

Geometry is one of the first established areas of mathematics, which has been known for its importance, as it provides foundational knowledge and helps to build the thinking skills. However, school geometry did not necessarily follow this diversity of geometry as a mathematics discipline (Jones, 2000). Firstly, the proportion of geometry has been reduced in many national curricula over the world in the past several decades (e.g., Backe-Neuwald, 2000; Glasnović Gracin & Kuzle, 2018; Mammana & Villani, 1998). One of the reasons for this was the temptation to increase the coverage of other mathematical disciplines within school mathematics, such as algebra, data analysis and probability (Jones, 2000). Furthermore, geometry curriculum has been criticized for the lack of coherence and for placing too much focus on terminology (Van de Walle & Lovin, 2006).

Empirical studies conducted in the last two decades also pointed to the current position of geometry in mathematics curriculum. Backe-Neuwald (2000) reported that primary mathematics teachers view geometry as secondary and less relevant than arithmetic and algebra, a sort of entertainment and relaxing part in comparison to the "more severe" mathematical contents with computations. Further on, the proportion of geometry items on international large-scale studies, such as PISA and TIMSS, is smaller in relation to items of other domains (e.g., Mullis & Martin, 2013; OECD, 2003). This information is valuable because the mentioned international studies may influence the structure and content of many national curricula (Volante, 2016). All these findings imply that geometry seems to have lost its central position in the mathematics curriculum. For that reason, it is not surprising that school geometry has been labelled as the "problem child" of mathematics teaching in recent decades (e.g., Backe-Neuwald, 2000).

These issues encouraged researchers and scholars to reassess the role of geometry within mathematics education and curricula with respect to perspectives on the geometry for the 21st century (Mammana & Villani, 1998). One of the approaches focuses on the idea of a well-established and coherent geometry curriculum by framing it in terms of “overarching ideas” (e.g., Van de Walle & Lovin, 2006) or fundamental ideas (e.g., Schweiger, 1992). This idea refers to a wider approach in mathematics education, providing both mathematics educators and students with several central themes that interconnect the different areas of mathematics and its applications. Also, such an approach enables valuable research on students’ acquisition of different fundamental ideas at different school levels and in different countries (e.g., Kuzle, Glasnović Gracin & Klunter, 2018; Kuzle 2019). However, the question of what fundamental ideas the students identify with geometry remains open. The work presented in this paper provides the first insight into the images Croatian elementary students have of geometry through the lenses of fundamental ideas. Such research has not yet been conducted in Croatia.

## Theoretical perspective

In this section, the construct of fundamental ideas as well as Wittmann’s model of fundamental ideas of geometry (Wittmann, 1999) are presented. Using drawings as a research method is then discussed.

### *Fundamental ideas of geometry*

According to Schweiger (1992), a fundamental idea may be described as a set of actions, strategies or techniques that (1) can be found in the historical development of mathematics, (2) appears viable to structure curriculum vertically, (3) seems suitable to talk about mathematics and answers the question what mathematics is, (4) makes mathematical teaching more flexible and transparent, and (5) possesses a corresponding linguistic or action-related archetype in everyday life. Fundamental ideas have a feature of a gradual and continuous development in every level of education (Rezat, Hattermann, & Peter-Koop, 2014; Van de Walle & Lovin, 2006), and therefore they represent axes along which competences may build up cumulatively. These features can be applied to geometrical fundamental ideas as well.

In line with that, Wittmann (1999) proposed that school geometry could be organized around the following seven fundamental ideas: (1) geometric forms and their construction, (2) operations with forms, (3) coordinates, (4) measurement, (5) patterns, (6) forms in the environment, and (7) geometrization. Description of these can be found in Table 1. Descriptions established by Wittmann (1999) are translated into English and further adjusted by Kuzle and Glasnović Gracin (in press). The Wittmann’s idea of fundamental ideas of geometry reflects diversity, coherence, and richness of geometry. Such *multi-dimensional* view of geometry is in line with the recommendations of the ICME-study for new geometry curricula (Mammana & Villani, 1998), and has been adopted by many international curricula (e.g., the United States, Germany).

*Table 1 Fundamental Ideas of Geometry with descriptions*

| Fundamental idea | Description |
|---|---|
| F1: Geometric forms and their construction | The structural framework of elementary geometric forms is three-dimensional space, which is populated by forms of different dimensions: 0-dimensional points, 1-dimensional lines, 2-dimensional shapes, and 3-dimensional solids. Geometric forms can be constructed or produced in a variety of ways through which their properties are imprinted. |
| F2: Operations with forms | Geometric forms can be operated on; they can be shifted (e.g., translation, rotation, mirroring), reduced or increased, projected onto a plane, shear, distorted, split into parts, combined with other figures and shapes to form more complex figures, and superimposed. In doing so, it is necessary to investigate spatial |

| | |
|---|---|
| | relationships and properties changed by each manipulation. |
| F3: Coordinates | Coordinate systems can be introduced on lines, surfaces and in space to describe the location of geometric forms with the help of coordinates. They also play an important role in the later representation of functions and in analytical geometry. |
| F4: Measurement | Each geometric form can be qualitatively and quantitatively described. Given units of measure, length, area or volume of geometric forms, and angles, can be measured. In addition, angle calculation, formulae for perimeter, area and volume, and trigonometric formulae also deal with measurement. |
| F5: Patterns | In geometry, there are many possibilities to relate points, lines, shapes, solids and their dimensions in such a way that geometric patterns emerge (e.g., frieze patterns). |
| F6: Forms in the environment | Real-world objects, operations on and with them, and relations between them can be described by using the geometric forms. |
| F7: Geometrization | Plane and spatial geometric facts, theorems and problems, but also a plethora of relationships between numbers (e.g., triangular numbers) can be translated into the language of geometry and described geometrically, and then translated again into practical solutions. Here, graph theory and descriptive geometry (e.g., parallel projection) play an important role. |

In the Croatian Syllabus for Primary Schools (Ministarstvo znanosti, obrazovanja i sporta [MZOS], 2006), five out of seven fundamental ideas are present (Glasnović Gracin & Kuzle, 2018), but they differ greatly in emphasis given to covered ideas. The emphasis in the primary level (grades 1 to 4) is put on only two fundamental ideas: *Geometric forms and their construction* and *Measurement* (Glasnović Gracin & Kuzle, 2018; 2019). The new curriculum (Ministarstvo znanosti i obrazovanja [MZO], 2019) comprises six fundamental ideas in compulsory education. In comparison to the old curriculum (MZOS, 2006), it includes additional fundamental idea of patterns (which concern geometric patterns as well). In comparison to other curricula for mathematics for primary levels, such as the one in Germany or in the TIMSS framework, axis symmetry and visual capacity are not included in the new Croatian curriculum for mathematics (Glasnović Gracin & Kuzle, 2019).

Glasnović Gracin and Kuzle (2018) conducted a case study in Croatia on four students' conceptions of geometry. The results showed that the images the participants had of geometry are strongly related to the fundamental idea *Geometric objects and their construction*, while the fundamental ideas *Operations with forms*, *Coordinates*, *Patterns*, and *Geometrisation* were underrepresented. Still, the sample was too small to develop a comprehensive picture of students' insights about geometry through the lenses of fundamental ideas.

*Using drawings as a research method*

A common method of investigating student activities in mathematics education is, among other methods, the use of questionnaires. Further on, the results are often obtained by direct observations and interviews with students. Still, these methods showed some shortcomings in studies with young children. Disadvantages of data collection through interviews and observation are a particularly large time expenditure. Also, disadvantages of questionnaires and interview surveys may be occasional unreliable answers provided by the students, due to, for example, their young age and the associated difficulty in expression (Ahtee, Pehkonen, Laine, Näveri, Hannula, & Tikkanen, 2016, p. 26). Pehkonen, Ahtee, Tikkanen and Laine (2011) also argue that young children may have problems in understanding the questions or the statements in questionnaires, and that this may lead to difficulties in understanding. Since the questions may seem rather irrelevant to children due to such difficulties in understanding, it is quite possible that the interviewer receives only short answers that do not provide meaningful information (Pehkonen et al., 2011).

A rather new approach to obtaining students' ideas and knowledge of teaching methods in mathematics education is the use and analysis of students' drawings. In the last two decades, this method has become more established, especially in studies with young children (Einarsdóttir, 2007; Ahtee et al., 2016). It is advantageous for data collection, interpretation and evaluation to obtain information about the students’ in-depth concepts. Since the students’ drawings mainly reflect non-verbal expression, the possible language barrier can be overcome and language mediation is hardly necessary (Ahtee et al., 2016). Barlow, Jolley and Hallam (2011) emphasized that the drawing process stimulates the child to talk about particularly relevant occurrences and events related to the situation depicted in the drawing. The process stimulates the child to remember certain incidents. The child may spontaneously reveal details about what has been produced. In this way, clues to certain situations that relate to what is drawn are provided. A question-answer scenario, unconscious to the child, may occur, in which the participant is asked to talk about the drawing in more detail. Even short expressions that are not relevant to the drawer, such as “Really?” or “Aha”, increase the amount of conversation about the drawing. In that manner, it provides an even deeper insight into the situation (Barlow et al., 2011).

Previous studies involving students’ drawings showed that they provide a complementary contribution to research in addition to the usual research methods, especially when working with young students. For example, using students’ drawings about mathematics teaching can provide useful information about the teacher, the classroom and classmates and provide a nonverbal impression of perceptions of (mathematics) teaching and learning (e.g., Ahtee et al., 2016; Pehkonen, Ahtee, & Laine, 2016). Nevertheless, using students’ drawings as a research tool in mathematics education is still not used to a greater extent.

### *Research questions*

The main goal of the study presented in this paper was to find the ideas that primary school students in Croatia have of geometry by using drawings. For this purpose, the following research questions were posed: What fundamental ideas of geometry can be seen in the primary school students’ drawings? How do students' images of geometry develop over the course of schooling?

## Methods

A qualitative research design was chosen for this explorative study. The study participants were Croatian primary students (grades 2-4, i.e. students aged 8 to 10 years). This age group was suitable for the study purposes as this is an important period for the development of geometric thinking (e.g., Mamanna & Villani, 1998). In total, 249 students from various Croatian elementary schools participated in the study (see Table 2).

*Table 2 Participant sample*

| Grade level | Grade 2 | Grade 3 | Grade 4 |
|---|---|---|---|
| Number of participants | 93 | 82 | 74 |

The main sources of data collection were students’ drawings and a semi-structured interview. Students’ work was based on an adaptation of the instrument originally designed by Halverscheid and Rolka (2006). The students were given a piece of paper with the following assignment: “Imagine that you are an artist. A good friend asks you what geometry is. Draw a picture in which you explain to him/her what geometry is for you. Be creative in your ideas.” In addition, the participants answered the following questions: “In what way is geometry included in your drawing?”, “Why did you choose these elements in your drawing?”, “Why did you choose this kind of representation?” and “Is there anything you did not draw but still want to say about geometry?”. The procedure was conducted in 2017 during the classroom practice by student teachers of The Faculty of Teacher Education in Zagreb, who had been trained to gather data and conduct interviews.

Data analysis involved coding the obtained data, and then validating the identified codes through an iterative process of constant comparison by the two authors. The data analysis encompassed the analysis of drawings according to the framework of Wittmann (1999), confirmation of the interpretation by content analysis of the questions posed, and coding other sub-conceptions included in the participants' additional answers. Different representations of fundamental ideas of geometry were firstly given one of the Wittmann's (1999) categories as presented in Table 1, and after that they were assigned a specific subcategory. Such analysis provided new subcodes, if a descriptor was not given before (for more information see Kuzle & Glasnović Gracin, in press). The final instrument used in the study is given in Table 3. The interrater reliability was high, and the rare disagreements were discussed and helped adjusting the coding in order for the final interrater reliability to be 100%.

*Table 3 Subcodes in a developed instrument to determine students' fundamental ideas of geometry*

| Fundamental idea | Subcodes |
|---|---|
| **F1 Geometric forms and their construction** | 0-dimensional objects (F1a), 1-dimensional objects (F1b), 2-dimensional objects (F1c), 3-dimensional objects (F1d), geometric properties (F1e), drawing and drawing/construction tool (F1f), non-geometrical tool for creating geometrical objects (F1g), angles (F1h), composite figures (F1i), plane and space (F1j) |
| **F2 Operations with forms** | translation (F2a), rotation (F2b), dilation (F2c), point symmetry (F2d), line symmetry (F2e), congruence (F2f), composing and decomposing (F2g), folding and unfolding (F2h), tessellation (F2i) |
| **F3 Coordinates, and spatial relationships and reasoning** | coordinate system (F3a), positional relationships (F3b), orientation and orientation tools (F3c), spatial visualization, relation and orientation (F3d) |
| **F4 Measurement** | length (F4a), perimeter (F4b), surface area (F4c), volume (F4d), angle measure (F4e), measuring tools (F4f), estimation (F4g), conversion of measuring units (F4h), scaling (F4i) |
| **F5 Geometric patterns** | Geometric patterns are created by using simple geometric forms (for example, concentric figure, or the seed of life). |
| **F6 Geometric forms in the environment** | Description/drawing of real-world objects, and operations on or with them using geometric forms/concepts. |
| **F7 Geometrisation** | geometrical facts (F7a), parallel projection (F7b), geometrical problems (F7c), figurate numbers (F7d) |

For example, Figure 1 presents an authentic participant's drawing about geometry. The participant drew four geometric solids – pyramid ("piramida"), cylinder ("valjak"), cube ("kocka") and a sphere ("kugla"). For that reason, each item got assigned F1d. Additionally, these solids are presented as real-world objects (pyramid in the desert, paper towels - "ubrus", dice - "kocka za 'Čovječe ne ljuti se'" and a watermelon - "lubenica"). As such, each item was assigned code F6. The drawing also includes two geometric tools: a ruler and a set square (triangle). Since these tools are not presented in the meaning of measuring, their assigned codes in this case were two F1f (drawing tool).

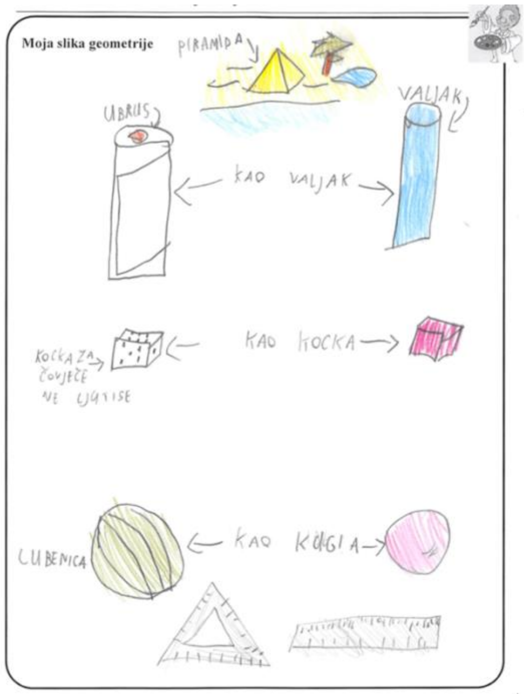


*Figure 1. A participant's drawing*

Each of 249 drawings was coded in this way. Each student’s work consisted of one or more coded items, as shown in the example in Figure 1. These data constituted a basis for calculation the descriptive statistics given in results: each code was given its relative frequency in relation to the amount of all coded items.

## Results

The results present which fundamental ideas of geometry can be seen in the primary students’ drawings and how these ideas differ at different school levels. These are presented here on the basis of the revised model of Wittmann’s fundamental ideas of geometry developed by Kuzle and Glasnović Gracin (see Table 3).

Table 4 shows the absolute and relative frequencies obtained in the participants’ drawings on their fundamental ideas of geometry. The fundamental idea *Geometric forms and their construction* (F1) was the most often illustrated fundamental idea of geometry (present in 88% of all coded items). This finding was independent of the grade level: almost all participants used at least one aspect regarding this idea. In Grade 2, participants drew geometric forms in 91% of all coded items, in Grade 3 it was 88% and in Grade 4 as much as 86% (Table 4). The fundamental idea *Geometric forms in the environment* (F6) was presented in 8% of all coded items (with 8% in Grade 2, 9% in Grade 3 and 6% in Grade 4). The fundamental ideas *Coordinates, and spatial relationships and reasoning* (F3), *Measurement* (F4) and *Geometric patterns* (F5) were minimally present in students’ drawings, while the fundamental ideas *Operations with forms* (F2) and *Geometrisation* (F7) were not present at all in Croatian primary students’ data.

*Table 4 Fundamental ideas of geometry in the students’ drawings*

| Grade | Absolute and relative frequencies of fundamental idea of geometry | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | F1 | F2 | F3 | F4 | F5 | F6 | F7 | Total |
| Gr. 2 | 569 (91%) | 0 (0%) | 2 (0%) | 1 (0%) | 2 (0%) | 53 (8%) | 0 (0%) | 627 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Gr. 3 | 553 (88%) | 0 (0%) | 2 (0%) | 16 (3%) | 5 (1%) | 54 (9%) | 0 (0%) | 630 |
| Gr. 4 | 499 (86%) | 1 (0%) | 10 (2%) | 32 (6%) | 3 (1%) | 33 (6%) | 0 (0%) | 578 |
| Total | 1621 (88%) | 1 (0%) | 14 (1%) | 49 (3%) | 10 (1%) | 140 (8%) | 0 (0%) | 1835 |

Considering the vertical examination, results show that no increase in amount or diversity is shown in students' drawings from Grades 2 to 4 with respect to different geometrical objects (F1). On the contrary, the proportions of coded items within the fundamental idea F1 slightly decrease from second to fourth grade. We notice the increase in the number of coded items assigned to fundamental idea F4 (Measurement) from grade 2 to Grade 4, from 0% to 6 %. Also, there were more occurrences of fundamental idea *Coordinates, and spatial relationships and reasoning* (F3) in Grade 4 than in lower grades, but they still present very small percentages. Proportions of coded items for other fundamental ideas, such as *Operations with forms* (F2), *Geometric patterns* (F5), *Geometric forms in the environment* (F6), and *Geometrisation* (F7), stay mainly unchanged at different grade levels (Table 4).

Since the fundamental idea *Geometric forms and their construction* (F1) was the most often presented one, it was reasonable to examine in more depth which aspects of this versatile idea were illustrated by the students. Each sub-code of the fundamental idea F1 was presented by relative frequency regarding all coded items within the particular school level. The results are shown in Table 5 (sub-codes *Non-geometrical tool for creating geometrical objects*, *Composite figures*, and *Plane and space* are omitted in Table 5 due to the fact that Croatian students barely presented these sub-codes in their drawings).

Findings indicate that in all examined grades, various 2-dimensional objects dominated in the participants' drawings, ranging from 47% in Grade 2, to 35% in Grade 3 and 34% in Grade 4. The second most often depicted aspect were lines and curves, which were used in approximately 20 % of all coded material. Similarly, geometric solids are present in 17% of all codes. Geometric properties with 5% in total and different Drawing tools with 4% in total were the next most often coded two aspects of F1 with slight increase from Grade 2 to 4. Points are illustrated in approximately 2% of all coded items. Angles are presented only in Grade 4, in 6 % of items.

*Table 5 Fundamental idea Geometric forms and their construction (F1) in students' drawings*

| Grade | Proportion of components of F1 within all codes | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0-dim. objects | 1-dim. objects | 2-dim. objects | 3-dim. objects | geom. properties | drawing tools | angles |
| Gr. 2 | 21 (3%) | 113 (18%) | 292 (47%) | 109 (17%) | 19 (3%) | 15 (2%) | 0 (0%) |
| Gr. 3 | 9 (1%) | 140 (22%) | 218 (35%) | 115 (18%) | 35 (6%) | 24 (4%) | 3 (0%) |
| Gr. 4 | 11 (2%) | 97 (17%) | 194 (34%) | 97 (17%) | 41 (7%) | 26 (4%) | 32 (6%) |
| Total | 41 (2%) | 350 (19%) | 704 (38%) | 321 (17%) | 95 (5%) | 65 (4%) | 35 (2%) |

With respect to the vertical examination, results show that participants mainly do not present more different geometric forms as they get older. In the case of 2-dimensional objects, which is the most frequent coded aspect of F1, we even notice a decrease in frequencies from Grade 2 to 4. It is also important to mention that these percentages differed most notably in comparison to any other vertical comparisons (47% in Grade 2 in relation to 35% and 34% in Grades 3 and 4). *Geometric* *properties* and *Drawing tools* are less presented in Grade 2 in comparison to Grades 3 and 4. Table 5 shows an increase in code F1h (angles), from 0% in Grade 2 to 6% in Grade 4.

## Conclusions and discussion

The results show that participating primary school students have a rather narrow perception of geometry with respect to diversity of fundamental ideas. Most of the students (more than 90% in each grade) represented either one or two fundamental ideas in their drawings (Table 6). Independent of the grade level, the fundamental idea *Geometric forms and their construction* (F1) strongly dominated in their drawings, while other ideas were much less presented, or not presented at all. This finding is in contrast with contemporary tendencies in understanding geometry as a versatile discipline (Wittmann, 1999; Mammama & Villani, 1998). The domination of the fundamental idea *Geometric forms and their construction* in comparison to other fundamental ideas raises the question on which curricular requirements are given in primary school geometry in Croatia. In line with the results, the insight into Croatian curricula shows that the emphasis is put on geometric shapes. Further on, the analysis of the curricula shows that geometry is underrepresented in topics, and at the same time within geometry curriculum there exists an apparent incoherence of geometric ideas, objects and relations (Glasnović Gracin & Kuzle, 2019; MZO 2018; MZOS, 2006). The Mathematics Syllabus for Primary Schools (MZOS, 2006) in grades 1 to 4 comprises only fundamental ideas *Geometric forms and their construction* (F1), *Measurement* (F4) and *Geometric forms in the environment* (F6), while the new Curriculum (MZO, 2019), along with the mentioned ideas, comprises also the idea *Geometric patterns* (F5). For example, according to the Syllabus for Primary Schools (MZO, 2006), in the third and fourth grade, the fundamental ides *Measurement* (F4) is taught, but the participants did not significantly use these ideas in their drawings. One of the reasons may be that the fundamental idea *Measurement* was hard for students to draw (Kuzle et al., 2018). The fundamental idea *Geometric forms in the environment* are presented in approximately 7% of codes in each grade. The fundamental ideas *Geometrisation*, *Coordinates, and spatial relationships and reasoning*, *Operations with forms* and *Geometric patterns* were not part of participants' curriculum (MZOS, 2006), and therefore it was expected for these ideas to be omitted in their picture of geometry.

Within the dominating fundamental idea *Geometric forms and their construction* (F1), student participants mainly represented 1-, 2- and 3- dimensional objects, with strong emphasis on plane shapes (almost 50% of all coded items in Grade 2). Within the geometry curriculum in Croatia (MZO, 2019; MZOS, 2006), the plane shapes (triangle, square, quadrilateral, disc) are emphasized in comparison to other sub-codes of *Geometric forms and their construction* (F1). The results show that the proportion of codes of 2-dimensional objects decrease from lower to upper grades, while geometric properties and angles slightly increase, which corresponds to learning new contents in third and fourth grades (MZO, 2019; MZOS, 2006). The previous results conducted by Kuzle et al. (2018), and Kuzle (2019) in Germany show similar emphasis on geometric objects by German primary school students. Still, German participants presented more drawings with Operations with forms (F2), which are not present at all in Croatian cases. Also, we notice bigger proportions in 1-dimensional objects presented in Croatian results. These findings require further study with comparative features and deeper comparison of two national curricula. To summarize, when looking through the lenses of fundamental idea framework, the results indicate a rather narrow view of geometry developed by the participants. Therefore, it would be necessary to re-question the curricular requirements regarding multi-dimensional nature of geometry and the coherence of its topics, as proposed by Mammana and Villani (1998).

The idea of using children's drawings to gain a better insight into what they understand under the term 'geometry' opened a new way going "beyond the purely cognitive". Such an approach might have a potential because some aspects are maybe easier to draw than to explain verbally (Ahtee et al., 2016; Pehkonen et al., 2011). Nevertheless, a few shortcomings are perceived: some participants had difficulties drawing, some do not like to draw in general, some rather drew objects which they found easier to illustrate, such as geometric shapes. It may be that the other aspects, such as measurement, were more difficult to draw. Additional data sources (e.g., post-interviews) were necessary here. In this sense, the search for alternative research methods, particularly in the studies with young children and in primary education, that would provide a holistic understanding of this multi-faceted phenomena, is an issue of concern, and remains an ongoing research area.

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# "Što je za tebe geometrija? Nacrtaj sliku." Kako učenici razredne nastave prikazuju geometriju kroz crteže


## Sažetak

Posljednjih nekoliko desetljeća smanjivao se udio geometrije u mnogim nacionalnim kurikulima diljem svijeta. Jedan od razloga leži u nastojanjima za povećanjem udjela drugih matematičkih disciplina unutar školskih matematičkih kurikula. Nadalje, geometrijski kurikuli su bili na meti kritika zbog nedostatka koherentnosti. Ove promjene ukazuju na pitanja o geometrijskim konceptima, strukturama i idejama koje učenici stječu na nastavi matematike. Ta pitanja su posebno važna za uzrast u razrednoj nastavi iz dvaju razloga. Prvo, jer se radi o vrlo važnom periodu za razvoj geometrijskog mišljenja. Drugo, te kompetencije čine bazu za kasnija usvajanja geometrijskih sadržaja. Istraživanje prikazano u ovom radu fokusirano je na shvaćanje geometrije kod učenika osnovne škole kroz analizu njihovih crteža. U istraživanju je sudjelovalo 249 učenika od 2. do 4. razreda osnovne škole iz Hrvatske. Sudionici su dobili prazan list papira i zadatak da nacrtaju što je geometrija, prema njihovom mišljenju. Analiza prikupljenih podataka provedena je uz korištenje adaptiranog Wittmanovog modela o sedam fundamentalnih ideja u geometriji kao teorijskog okvira. Rezultati pokazuju da učenici u primarnom obrazovanju imaju prilično usku sliku o tome što je geometrija. Na crtežima je dominirala fundamentalna ideja geometrijskih oblika, bez obzira na razred, dok su ostale fundamentalne ideje bile minimalno prisutne. Ovi rezultati ukazuju na dodatna promišljanja školskog kurikula za matematiku vezano uz višedimenzionalnost i raznolikost geometrije.



***Ključne riječi****: crteži; fundamentalne ideje; nastava geometrije; primarno obrazovanje*

# The development of the number sense through the comparison of the national, Montessori and Waldorf curriculum for the primary mathematics education in Croatia

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## Abstract

Recommendation from the Council of Europe on the key competences for lifelong learning features mathematical competence that builds "on a sound mastery of numeracy" and includes "sound knowledge of numbers". Numbers are an important part of primary mathematics education and they permeate many aspects of professional and social life.
The notion *number sense* motivated studies with a different focus and approach. Educators examined how number sense correlated with achievement and abilities in mathematics and investigated and designed activities to develop students' number sense. Number sense concerns "understandings, skills and attitudes about number" that are beyond being numerate in the everyday workaround mathematics.
The achievement of the Croatian pupils in the international surveys and national exams and the prospective primary school teachers' answers in a test developed by the author indicate low-level abilities in the working with numbers. Having experience with the alternative education, Montessori and Waldorf, we questioned how the corresponding curriculums promote the development of the number sense in the context of the primary mathematics education in Croatia.
We created a tool for curriculum analysis that included ten common components of the numbers sense from the literature. Our results showed that the Croatian curriculum failed by comparison in the number sense components related to the size of numbers, patterns and attitudes, and the outcomes which pertain to the components related to the relationship, representation, calculation and application, are rigid, limited and routine. We believe that educating prospective and practising teachers about alternative approaches to mathematics education might promote teaching and learning to develop the number sense.

***Key words**: curriculum analysis; primary school mathematics; Montessori education; Waldorf education*

## Introduction

Recommendation from the Council of Europe on the key competences for lifelong learning (2018) features mathematical competence that builds "on a sound mastery of numeracy" and includes "sound knowledge of numbers". The frameworks for international PISA and TIMSS surveys and Croatian national curriculum comprise the knowledge of numbers. Numbers are an important part of primary mathematics education, and they permeate many aspects of professional and social life.

The notion *number sense* has motivated studies with a different focus and approach. The problematic of the studies varied from the attempts to define the notion or to parse number sense into distinct components; all to the research about students' competences related to whatever it is implied by number sense. Understanding number sense and promoting its development is essential for early mathematics education and lifelong competences.

In this paper, we examined four mathematics curriculums: two Croatian curriculums for primary education, and Waldorf and Montessori recommendations for teaching and learning mathematics for 6-10 years old pupils. Even though these documents do not give a complete picture of mathematics education, they provide an insight into what each of the pedagogies or policies deemed relevant for compulsory mathematics education.

## Literature review

### *The elusive definition of number sense*

The notion of "number sense" might appear as a common-sense phrase that is used to describe a person's competences in the work with numbers. In education research, the notion is not well defined, that is, the researchers have different ideas on what is understood by the phrase number sense.

The first distinction was made between the preverbal quantity sense and computation-based number sense (Andrews & Sayers, 2015; Wagner & Davis, 2010). Psychologists are interested in the preverbal number sense as it corresponds to competences innate to all humans and to some extent to other species. Number sense refers to the understanding of numbers that humans acquire by instructions and develop by manipulation. Number sense applies across all ages and permeates different areas of mathematics and everyday personal, social and professional life (Howell & Kemp, 2010; Mcintosh et al., 1992).

Dunphy (2007) found that number sense is a difficult-to-define notion. Greeno (1991) observed number sense as situated knowing in a "conceptual environment of numbers and quantities". For him, developing number sense is about working in that environment and developing expertise by interacting with other individuals and resources from the environment. Mcintosh et al. (1992) provided the following definition:

> *"Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity. " (p.3)*

### *Components of number sense*

Greeno (1991) suggested that number sense should be theoretically examined in order to determine and understand its underlying properties. In his work, Greeno stated some conjectures about number sense. A person should be able to make quantitative judgments and inferences, be familiar with the notation and perform flexible computations and computational estimation. The situated cognition model that Greeno used alludes to the social aspect of number sense as part of the reasoning and learning. Finally, he elaborated on the mental models of numbers and quantities and how these influence number sense.

Mcintosh et al. (1992) addressed number sense as an acquisition required for every person from their compulsory education. They provided an elaborate framework with three major categories of number sense: (1) the knowledge of and facility with numbers, (2) the knowledge of facility with operations and (3) applying knowledge of and facility with numbers and operations to computational settings. The framework was further accessed and Reys et al. (1999) identified six major content strands, two from each of the three categories of the original framework. We present the components of number sense found in the frameworks in Table 1.

Dunphy (2007) designed a framework for number sense of four years old children. Apart from the analysis of the literature on number sense, she considered the theoretical perspectives in pedagogy, learning and assessment relevant for early childhood. Howell and Kemp (2010) investigated number sense skills of preschool children. Their framework was based on literature review and a Delphy study with mathematics education researchers. That was a two-step anonymous study where participants expressed their opinion about what components are indicative of number sense. Andrews and Sayers (2015) used the term foundational number sense (FONS) to acknowledge the competences children acquire in the first years of their mathematics education and that are fundamental for the mathematical competence.

Yang and his associates examined the competences of students in Taiwan at different levels of their mathematics education (Yang et al., 2009; Yang & Li, 2008; Yang & Wu, 2010). They used items (tasks, questions) that correspond to the following components of number sense:

- understanding the basic meaning of numbers and operations,
- recognizing relative and absolute magnitude of numbers,
- being able to compose and decompose numbers,
- being able to use a benchmark appropriately,
- recognising the relative effect of operation on numbers,
- using appropriate and flexible strategies to solve numerical problems, including estimation, mental computation, and so on, and judging the reasonableness of computational results.

*Table 1 The components of number sense according to the frameworks from the literature*

| | |
|---|---|
| **Andrews and Sayers (2015)** | Number recognition<br>Systematic counting<br>Awareness of the relationship between number and quantity<br>Quantity discrimination<br>An understanding of different representations of number<br>Estimation<br>Simple arithmetic competence<br>Awareness of number patterns |
| **Dunphy (2007)** | Pleasure and interest in number<br>Understandings of some of the purposes of number<br>Quantitative thinking<br>Awareness and understanding of written numerals |
| **Howell and Kemp (2010)** | Counting<br>Number principles<br>Number magnitude |
| **Reys et al. (1999)** | Understanding of the meaning and size of numbers<br>Understanding and use of equivalent representations of numbers<br>Understanding the meaning and effect of operations<br>Understanding and use of equivalent expressions<br>Flexible computing and counting strategies for mental computation, written computation, and calculator use<br>Measurement benchmarks |

*Students' competences related to number sense*

Number sense was found to be a prerequisite for developing mathematics abilities and a predictor of success in mathematics (Andrews & Sayers, 2015; Howell & Kemp, 2010). Research has been conducted with respect to (different components of) number sense at all educational levels. Studies showed unsatisfactory students' performance and indicated some misconceptions students hold about numbers (Yang & Li, 2008). Students chose rule-based and practiced strategies over the strategies grounded on number sense (Almeida et al., 2016; Yang et al., 2009; Yang & Wu, 2010).

Questioning students' performance in different components of number sense informed about issues in mathematics curriculum and teaching that obstruct or impede the development of number sense (Howell & Kemp, 2010). For example, preschool mathematics education needs to place emphasis on the principles of counting, the number conservation, order irrelevance, ordinality and cardinality.

Some of the studies related to number sense were experimental. Teaching, discussing and sharing number-sense based strategies had positive impact on students' results in the post intervention and retention tests in studies by Markovits and Sowder (1994) and Yang, Hsu and Huang (2004).

Number sense is a competence that enables a person to effectively, flexibly and reasonably manipulate numbers and operations to resolve issues in mathematical, social, private and professional situations. It depends on purposeful instruction and therefore should be an essential part of mathematics curriculum.

## Context of the study

Croatian pupils participated in TIMSS surveys in 2011 and 2015. The first time, their average result was under the TIMSS average and the second time the average result in the domain of numbers was lower than the average result in other domains. The results in the domain of quantity in PISA 2012 survey, although higher than in other domains, was below the OECD average. According to the qualitative analysis of the national exams conducted in 2008, ten years old pupils mainly used the written computation and performed significantly lower in multiplication.

The author created a set of tasks to address computational skills and recognition of real number magnitude of the prospective primary school teachers. The remnants of rule-based procedures and rote algorithms were observed. For example, students calculated the sum of fractions 1+1/2 using a common denominator; they determined 25% of the number 1320 by written multiplication of numbers 1320 and 0.25; they determined the unknown in the expression 5/7=20/(x+1) by solving corresponding equation, etc.

Number sense is an integral part of national curriculums worldwide. Research, however, suggests that textbooks and teachers' practice do not follow that recommendation (Dunphy, 2007; Howell & Kemp, 2005; Yang & Li, 2008; Yang & Wu, 2010). The reason could be that there is no general understanding about what number sense is and how to develop it. On the other hand, there are no studies of number sense in the context of Croatian mathematics education. Our experience with alternative pedagogies, Montessori and Waldorf, motivated us to explore the following:

How does the Croatian national curriculum correspond to different components of number sense?

What indications do the Croatian national curriculum and the recommendations for the Waldorf and Montessori mathematics education provide for the development of number sense?

How does the Croatian national curriculum compare with the recommendations for the Waldorf and Montessori mathematics education with respect to the components of number sense?

## Methods

The literature review provided several frameworks to explain number sense. We created a cumulative analytical tool that included ten components of number sense (Table 2). Each of the components was grounded on some distinct property of number sense found in the frameworks examined in the literature. For example, the component labelled "Attitude" refers to the social aspect of knowing numbers, as Greeno described it, and the component "Pleasure and interest in number" from Dunphy's framework. Some components in our analytical tool cover several properties of number sense recognized in the literature. For example, the component labelled "Magnitude" includes the correspondence between a number and a quantity, the relative and absolute size of a number and the system of benchmarks, which were mentioned individually in several frameworks.

*Table 2 The components of the designed tool for number sense*

| Component | The properties of the component of the designed tool for comparison |
|---|---|
| COUNTING | Counting skilfully and flexibly<br>Mastering the principles of counting<br>Making judgments based on counting |
| SYMBOLS | Knowing and using the numerals and terminology |
| RELATIONSHIP | Determining the relationship between numbers and quantities<br>Making judgments about the relationships between numbers and quantities |
| MAGNITUDE | Establishing the correspondence between numbers and quantity<br>Understanding the relative and absolute size of a number and quantity<br>Estimating and using the system of benchmarks<br>Making judgments about quantity |
| REPRESENTATION | Knowing, using and converting different representations of a number<br>Composing and decomposing numbers and quantities<br>Making judgments about the representations of a number |
| COMPUTATION | Computing skilfully and flexibly<br>Understanding the meaning of operations<br>Knowing and using the properties of operations |
| CHANGE | Understanding the effect of operations on numbers<br>Estimating and judging the result of operations |
| USEFULNESS | Developing and applying appropriate and efficient strategies of using numbers and operations for solving problems |
| PATTERNS | Recognizing, expanding, maintaining or constructing a sequence or pattern of numbers |
| ATTITUDE | Having interest in numbers and operations |

We examined two curriculum documents for primary mathematics education in Croatia (Ministarstvo znanosti i obrazovanja, 2018; Ministarstvo znanosti, obrazovanja i športa, 2006), the curriculum for Waldorf school and a handbook for Montessori education. From those documents we selected the outcomes, content or instructions related to numbers and operations and itemized them accordingly. Each item was arranged in one or more cells of a component of number sense versus recommended target age matrix. The methodology is based on the *topic trace mapping* that was used for the TIMSS curriculum analysis (Schmidt, 1992).

This representation enabled us to observe which components were present and emphasised in each curriculum, what timing and duration is recommended to develop each component, and how the traditional and alternative approaches compare with respect to number sense.

## Results

### *The outcomes of the Croatian curriculum against the components of number sense*

There is a significant difference in the 2006 and the 2018 mathematics curriculum. The former is a detailed teaching programme and outcomes were broken apart according to the sequential teaching topics. For example, the 2018 curriculum stated a child in the first grade should *add and subtract in the set of numbers up to 20*; whereas the 2006 curriculum separated the same outcome into nine different statements.

Observing the 2006 curriculum, working with numbers is a major topic in first three grades (Table 3). The focus is on the component labelled *computation*, thus including knowing computation facts and procedures of written computation and using properties of and relationships between operations. The component labelled *symbols* has a significant part in the curriculum for the first and second grade. Herein, the list contains reading and writing ordinal and cardinal numbers and Roman numerals, writing symbols for relationship and operations, using terminology in operations (addends, sum, minuend, subtrahend and difference).

*Table 3 The frequencies of correspondence between the statements in topics from the 2006 mathematics curriculum and the components of number sense*

| Component | First grade 15 of 21 topics | Second grade 28 of 31 topics | Third grade 14 of 23 topics | Fourth grade 9 of 22 topics |
|---|---|---|---|---|
| COUNTING | 3 | 1 | 1 | 1 |
| SYMBOLS | 11 | 7 | 1 | 1 |
| RELATIONSHIP | 3 | 1 | 1 | 1 |
| MAGNITUDE | | - | - | - |
| REPRESENTATION | 5 | 3 | 1 | 1 |
| COMPUTATION | 9 | 19 | 15 | 8 |
| CHANGE | - | 7 | - | - |
| USEFULNESS | 1 | - | - | - |
| PATTERNS | 1 | 2 | - | - |
| ATTITUDE | - | - | - | - |
| Number of statements | 33 | 40 | 19 | 12 |

Comparing numbers and expressing the relationship between numbers pertain to the component labelled *relationship* throughout grades. The component labelled *representation* includes displaying numbers and operations on number line, differing between cardinal and ordinal numbers and displaying multi-digit number as the sum of place-values. In the second grade, statements significant for the component labelled *change* refer to properties of multiplication, a relationship between multiplication and division, and the role of numbers one and zero in multiplication and division, and use the relations.

The rest of the statements are: (1-21) knowing the procedure of solving word problems in the component *usefulness*, (1-13) understanding the formation of sequence of whole numbers, (2-3) marking with ordinal number a term in a sequence and (2-30) differentiating between odd and even numbers in the component *patterns*.

We extended the analysis of the 2018 curriculum to the elaboration of the outcomes given in the document (Table 4). For that reason, a single outcome would be assorted into several components in our analytical tool. An essential outcome related to numbers would be using numbers to represent quantity and order, which includes counting, writing and reading numbers, comparing numbers, recognizing the value of a number and decimal values and displaying numbers in different forms. This outcome covers through the components *counting*, *symbols*, *relationship*, *quantity* and *representation*. In the first two grades, additional focus is on the ordinal and cardinal numbers and Roman numerals. Related to the outcomes within the components labelled *relationship* and *representation*, the curriculum recommendations emphasize the necessity of using concrete models in the first two grades and the importance of decomposition of a number using its place-values in the third and fourth grade.

*Table 4 The outcomes from the "School for life" curriculum against the components of number sense*

| Component | First grade | Second grade | Third grade | Fourth grade |
|---|---|---|---|---|
| COUNTING | A1 | A1 | A1 | A1 |
| SYMBOLS | A1, A2, A3, A4/B1 | A1, A2, A4 | A1, A2, A5 | A1, A2, A4 |
| RELATIONSHIP | A2 | A1 | A1 | A1 |
| MAGNITUDE | A1 | A1 | A1 | A1 |
| REPRESENTATION | A1, A3 | A1 | A1, A2 | A1 |
| COMPUTATION | B1/A4 | A3, A4, A5 | A2, A3, A4, A5 | A2, A3, A4 |
| CHANGE | - | A3, A4 | A2, A3 | A3, A4 |
| USEFULNESS | A5 | A3, A4, A5, A6 | A2, A3, A5, A6 | A2, A3, A4 |
| PATTERNS | A3, B2 | A4, B1 | - | - |
| ATTITUDE | - | - | - | - |

The elaborations of outcomes related to the component labelled *computation* include statements related to other components of the framework (Table 5). In the second grade, the elaboration of the outcome A.2.4 includes the understanding effects of the multiplication and division on numbers that is a feature of the component labelled *change*. Additionally, attention is paid to using properties of and relationship between operations, mental computation, estimation and assessment of the result of a computation which correspond to the components labelled *computation* and *change*. In the first two grades, outcomes B.1.2 and B.2.1 for recognizing a pattern and expanding a sequence belong to the component labelled *patterns*. The outcomes A.1.5, A.2.6, A.3.6 and A.4.4 correspond to applying number and operations in solving tasks and everyday problems, which is relevant for the component labelled *usefulness*.

*Table 5 The elaboration of an outcome from the "School for life" curriculum*

| Outcome | A.3.2 “adds and subtracts in the set of whole numbers up to 1,000” |
|---|---|
| SYMBOLS | naming the terms in a computation |
| USEFULNESS | solving word problems |
| COMPUTATION | mental addition and subtraction of numbers up to 1,000<br>using the commutative property and the relationship between addition and subtraction<br>using appropriate mathematical record of written addition and multiplication |
| CHANGE | estimating the result of addition and subtraction |
| REPRESENTATION | determining place-value of digits in a three-digit number |

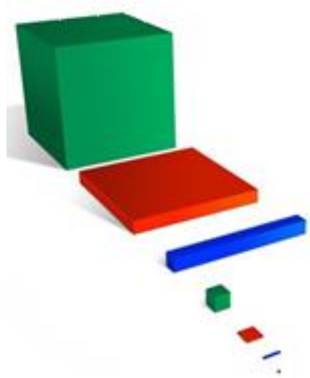
### *The promotion of number sense in the Montessori and Waldorf curriculums*

The two alternative pedagogies have an outstanding philosophy of education (Edwards, 2002). Rudolf Steiner founded the Waldorf education a hundred years ago to contrast the academic, formative education. The Waldorf curriculum is systematically shaped to address, vertically and horizontally, an individual’s cognitive, artistic, physical and social development. Teaching mathematics is a part of a holistic approach to education, as it connects a child’s temper and movement into learning through rhythm. The Montessori curriculum places a child in the centre of the education process under the slogan “help me to do it myself”. The role of the teacher is to observe each individual child and to prepare the wholesome learning environment. Montessori manipulative activities have control of error.

Both pedagogies introduce numbers, counting and quantities early in education, as early as three years of age. In the Waldorf curriculum, counting is modelled with songs, dance and physical polygons. A number wheel has ten pegs labelled with decimal digits. One skip counts and wraps yarn around a peg which corresponds to the last digit of the number (Figure 1). The accent is on the qualitative rather than the quantitative aspect of numbers. Children recognize quantity of a number through themselves and their environment. For example, they state there is one of me, one Sun, one Moon but two eyes, two ears and two parts of the day. Waldorf accentuates the attitude towards numbers and operations. Each operation has its own anthropomorphised temper. For example, a dwarf called Minusko is blue, he is choleric and he always gives away (rather than takes away).

In the Montessori curriculum, children have access to different manipulative activities to model numbers, operations and relationship between them. The first encounter with counting and numbers is with red-blue rods. Number one is a red, one decimetre long rod; number two is red-blue, 2 decimetre long rod, etc. A child makes judgment about the relationship between numbers while referring to the rods. For example, number four is larger than number two because one can make rod of four with two rods of two. Later on, each number from one to ten has its own assigned colour. Montessori manipulative enables development of an attitude about the relative size of numbers. For example, we observe one versus million with blocks as in Figure 1.

A skip counting wheel from Waldorf



A hierarchy of numbers from Montessori

*Figure 1.*

It is beyond the scope of this paper to represent many examples of activities from the Waldorf and Montessori curriculums for the development of the numbers sense. We will reflect on the "understanding the effect of operations on numbers" from the aspect of Waldorf and Montessori education (Figure 2). This statement is a part of the component labelled *change,* which is lacking in the Croatian curriculum. In Waldorf, first-graders work with all four operations in the set of integers up to twenty. Therefore, children relate quantity to the result of different operations to realize how each operation changes the numbers.

In Montessori, children interpret numbers and computations with manipulative whereby they observe the effect of operations by connecting different representations. From the age of six, children use the chain of multiples to represent and relate squares and cubes of integer numbers up to ten.


A holistic approach to computations from Waldorf

A representation of computations from Montessori

*Figure 2. Approaches to understanding the effect of operations on numbers*

## Discussion

The analysis of the outcomes in the 2006 curriculum revealed the focus was on memorizing facts and performing procedures. Some of the statements were unclear; they appeared random and incoherent across grades. From that aspect, we observed a major shift in the 2018 curriculum toward developing conceptual along with procedural knowledge and maintaining a vertical correlation of the outcomes. What makes a difference is that the 2018 curriculum emphasizes displaying a number in different forms in the component labelled *representation,* discussing the value of a number and decimal units in *magnitude*, mental *computation*, estimation and assessment of computation in *change*, and using numbers and operations to solve problems in the component labelled *usefulness*.

Both the 2006 and 2018 curriculum showed lack of outcomes that correspond to the components labelled *pattern* and *attitude* and to some distinct features of the components labelled *relationship, magnitude,* and *change*. Firstly, the curriculums focus on comparing and ordering numbers. Secondly, we found no indication of understanding the relative and absolute size of a number and quantity or estimating and using the system of benchmarks. Thirdly, though the 2018 curriculum provides an estimation and assessment of computation, both curriculums include only some aspects of the effect of operation on numbers, i.e. the effect of multiplication.

The analytical tool based on topic trace mapping consisting of ten components that describe number sense was applied in this study. Deliberating the assortment of the outcomes raised a few questions about the components of the framework:

What is the difference between the *relationship* and *magnitude*?

How are the features of the component *change* manifested?

Should the components *patterns* and *attitude* be a part of the framework?

The framework provides a distinction between the *relationship* and *magnitude*. The difference might not be that obvious in the educational practice. Outcomes and activities can contribute to both components, for example recognizing the relationship between decimal units from the 2018 curriculum. The features of the component *magnitude* relate to the one-to-one correspondence between the number and the quantity it represents and having a conceptual idea about that particular relationship.

Knowing the effect of operations on numbers enables one to make meaningful and reasonable estimates and assessments of computation. The acquisition of the feature 'knowing the effect of operations' of the component labelled *change* is absent from the curriculum. It is possible to develop it through modelling and representing the operations. We exemplified the aspects of this component with the recommendations from the Montessori and Waldorf curriculums.

Recognizing patterns is important for developing the principles of counting, working with the multi-digit numbers, identifying multiples, etc. Though it is an essential mathematical competence, the literature review did not often claim it indicative of number sense. Further, in the Croatian mathematics framework it is assorted to the domain Algebra rather than domain Numbers. An attitude toward numbers and operations was also rarely found in the literature review. However, we believe that without a positive attitude, self-engagement and intention there could be no advancement in developing one's number sense.

## Conclusion

In order to access number sense we need to clarify what number sense is, how it is manifested and how to develop it (Howell & Kemp, 2005). In this paper, based on the literature review, we described number sense as manipulating numbers and operations in various situations. The manipulations ought to be effective, flexible and reasonable. We acknowledged the components that described number sense and confronted them to the Croatian mathematics curriculum for primary school. The components labelled *counting*, *symbols*, *relationship* and *magnitude*, *representation*, *computation*, *change*, *usefulness* and *attitude* provide a framework for describing number sense, at least in the context of primary mathematics education in Croatia. We provided examples of activities characteristic of alternative pedagogies that promote the development of each of the components of the framework.

The nuances in the theoretical approach to number sense might raise awareness about some aspects of working with numbers and computation that might not be evident in the educational practice. The new curriculum switched focus to some aspects of flexible computation and the representation of numbers, but other theoretical aspects of number sense elude it.

Developing number sense should be an educational goal. The availability of technology and accessibility of quantitative information diminish the knowledge of computational procedures and demand interpretation and judgment. In order to develop flexible number sense, teaching should also be flexible. We believe that educating prospective and practising teachers about alternative approaches to mathematics education might well promote teaching and learning to develop the number sense. Effective, flexible and reasonable teaching approach contributes to developing likewise (teachers' and) pupils' number sense strategies.

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# Razvijanje “osjećaja za broj” prema usporedbi nacionalnog, Montessori i Waldorfskog kurikuluma za matematiku u nižim razredima osnovne škole u Hrvatskoj


## Sažetak

Preporuke Vijeća Europe o ključnim kompetencijama za cjeloživotno obrazovanje uključuju matematičku kompetenciju koja se temelji na sposobnostima računanja i znanju o brojevima. Teorijski okviri međunarodnih istraživanja PISA i TIMSS te hrvatski nacionalni kurikulu sadrže komponentu rada s brojevima. Brojevi su važan dio matematičkog obrazovanja u nižim razredima osnovne škole i neizostavni su u brojnim aspektima profesionalnog i društvenog života.

Pojam osjećaj za broj (eng. *number sense*) motivira istraživanja s različitim fokusom i pristupom. Psiholozima je zanimljiv urođeni osjećaj za broj. Metodičari su ispitivali kako je osjećaj za broj povezan s uspjehom i sposobnostima u matematici te istraživali i dizajnirali aktivnosti koje razvijaju osjećaj za broj. Dunphy (2007) je pisao kako je osjećaj za broj pojam koji se teško definira. Osjećaj za broj odnosi se na razumijevanja, vještine i stavove o broju koji nadilaze sposobnosti rada s brojevima u svakodnevnim situacijama.

Rezultati hrvatskih učenika na međunarodnim istraživanjima i nacionalnim ispitima te odgovori studenata učiteljskih studija na testu koji je osmislila autorica, ukazuju na slabije sposobnosti u radu s brojevima. Iskustvo autora s alternativnim obrazovnim sustavima, Montessori i Waldorf, potaknulo je ispitivanje kako pojedini kurikuli podržavaju razvijanje osjećaja za broj u kontekstu matematičkog obrazovanja u nižim razredima osnovne škole u Hrvatskoj.

Pregledom znanstvene literature pronađeno je nekoliko teorijskih okvira koji tumače pojam osjećaja za broj (Andrews i Sayers, 2015; Dunphy, 2007; Howell i Kemp, 2010; Yang, 2005). Uspostavljen je okvir za usporedbu kurikula koji se sastoji od deset prepoznatih komponenti osjećaja za broj. Rezultati pokazuju kako hrvatski kurikulu u usporedbi s alternativnim kurikulima podbacuje u komponentama povezanima s veličinom broja, uzorcima i stavovima o broju. Nadalje, ishodi učenja u hrvatskom kurikulu, a koji pripadaju komponentama povezanima s odnosima, prikazima, računanjem i primjenom broja, nefleksibilni su, ograničeni i rutinski. Razvijanje osjećaja za broj, u odnosu na uvježbavanje procedura, može se ostvariti aktivnostima važnima za komponente osjećaja za broj, a podzastupljenima u hrvatskom kurikulu. Smatramo kako obrazovanje budućih učitelja i učitelja u praksi o alternativnim pristupima matematičkom obrazovanju može doprinijeti učenju i poučavanju koje razvija osjećaj za broj.


***Ključne riječi****: Montessori obrazovanje; osjećaj za broj (eng. number sense); osnovnoškolska matematika; usporedba kurikula; Waldorfsko obrazovanje*

# (To What Extent) Can "Mathematical Literacy" Be Implemented Sustainably Through Centralized Secondary School-Leaving Examinations? Some Insights from Austria

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## Abstract

In 2009, the Austrian National Council decided that centralized school leaving examinations ("Zentralmatura") should be introduced at all secondary schools in the future. The framework concept for the final examinations in mathematics at academic secondary schools (AHS) was co-developed by mathematics educators, and its educational objectives are based on the concept of mathematical general education ("Allgemeinbildung") that bears many traits of a "mathematical literacy" conception.
Centralized school-leaving examinations in mathematics at academic secondary schools have been taking place for five years now, and from the outset and throughout these have been subject to massive scrutiny by teachers, parent representatives, mathematicians, and the media.
This paper examines which conclusions can be drawn from the experiences of the past five years, regarding a sustainable implementation of ambitious conceptions such as "mathematical literacy" by means of "new governance" such as centralized final examinations.

***Key words:*** *Allgemeinbildung; educational governance; standardized testing*

## In lieu of an introduction: A (not entirely) fictional narrative rationalization

Asking to what extent, if at all, "mathematical literacy" can be implemented sustainably trough centralized secondary school-leaving examinations already presupposes that both of them – "mathematical literacy" and centralized examinations – have something to do with each other in the first place. In lieu of a proper introduction, I will justify this assumption in a bit of an odd fashion: a (not entirely) fictional twelve-step narrative rationalization for promoting "mathematical literacy" by means of centralized examinations.

1. Most of us can possibly agree that compulsory mathematics education can and should matter to all students, not just to those who take a spontaneous liking to it or those who aim for STEM careers.
2. Some have argued that in modern societies one can hardly become a well-informed citizen or employee without some (basic) understanding of mathematics and statistics and its "formatting power" (Skovsmose, 1994) on society, economy and culture – which is the gist of mathematical literacy.
3. A curriculum that takes "mathematical literacy" seriously needs to shift its perspective: most adults will be consumers, rather than producers of mathematical models or statistical information (c. f. Fischer, 2001; Gal, 2002).
4. Contemporary mathematics instruction pays too much attention to the mastery of relatively elaborated computational skills and artificial, contextless problems (Peschek, 2009), which is, after all, only really good for producers/specialists of mathematics and does a rather poor job in engaging students in societal or even vocationally relevant mathematics (which would be relevant to consumers of/laypersons in mathematics).
5. This needs to change.

6. No really, we are quite serious: Just look at our miserable PISA results (Pöll, 2010)! Even Germany nearly overtook us!
7. But educational systems are so sluggish, and teachers and students are so stubborn – if we want something to change, we need to raise the stakes.
8. What ultimately matters to students and teachers alike are the exams, so let us govern them by introducing centralized school-leaving examinations. If we change those exams, mathematics instruction will surely follow suit.
9. Let's not beat around the bush: It is hard to develop good items which conform to our ideals of mathematical literacy and help to communicate them to teachers, students and parents, while also letting at least 90% of students pass the exams.
10. We are so sorry, but we missed the mark again: 20% failure rate, school teachers are angry, because of all those helicopter parents bothering them. Let's lower the bar a bit next time then, shall we?
11. Thank goodness we are back at 10% failure rate! But look: it's those pesky university STEM professors writing another letter of appeal (Taschwer, 2018), because the state is cutting their funding due to their notoriously high failure rates in introductory math courses and they all blame it on our school reforms. Maybe it would be better to include some more procedurally complex items next time then, shall we?
12. Go back to step 10, repeat, ad infinitum.

## The case of Austria – some factual corrections

Any jokes aside, the above rationalization is at least somewhat based on the realities of centralized school-leaving examinations in Austria (often referred to by their colloquial Austrian shorthand "Zentralmatura"). In some parts some rather important factual corrections seem in order: although it is a popular misconception, "inventing" or even advocating for such centralized examinations cannot be attributed to the Austrian mathematics education research community. It was simply a governmental decision dating back to the year 2009. When the government introduced "Zentralmatura", this was initially motivated by *increasing the quality, comparability and fairness of examinations*. There was no explicit mention of "mathematical literacy" or related constructs more closely related to German speaking traditions ("Bildung", "Allgemeinbildung" or "Grundbildung", c. f.). In fact, such centralized examinations have to be seen in the context of other measures of increasing accountability within the education system usually summed up under the label *educational governance* (or "Neue Steuerung", c. f. Klein, 2013; Kahnert et al., 2015), which has very little to do with the emancipatory aims usually associated with conceptions like "mathematical literacy" or "Allgemeinbildung".

To the contrary, educational governance has a lot do with the larger neo-liberal program (c.f. Moos, 2017). As Colin Crouch (2016) argued, the wide adoption of performance indicators for different public services (including public schools) constitutes an "attempt to make public services behave as though they exist in free markets" (Crouch, 2016, p. 1). These public services usually do not compete for customers and are not profit oriented. Performance indicators than act as a kind of "replacement currency" public services compete for while hopefully increasing the quality of the respective service in the process. Crouch (2016) gives some very compelling examples how systems of performance indicators are "gambled" by public institutions, i.e. how public institutions try to artificially increase the numbers when in fact the quality of the service to be evaluated does not change all that much. The quality of the service might even get worse because performance indicators always have to be restricted to what can be assessed in an easy manner. But what is measurable in a convenient manner might not always be what really matters for the quality of a service. Although at the moment the results individual schools achieve in the national centralized school-leaving examinations do not get published, there has been an ongoing discussion to exactly do this and it has been justified with the corresponding rationale that parents should make informed decisions about which schools their children attend and, in turn, schools should compete for their students to increase overall quality of schooling (Bayrhammer, 2016).

As far as the Department of Mathematics Education – Austrian National Competence Centre for Education in Mathematics at the University of Klagenfurt is concerned, it was responsible for a pilot project carried out from 2008 to 2012 for the Zentralmatura of Academic Secondary Schools (AHS), under the direction of Werner Peschek and with Roland Fischer participating as a member of the project team. It is correct that this pilot project was heavily influenced by theories of “Allgemeinbildung” (translated roughly: general education), especially Roland Fischer’s own theory of “höhere Allgemeinbildung” (higher general education), which can be related closely to emancipatory conceptions of “mathematical literacy” (c.f. Biehler, 2019; Vohns, 2017). But some considerable limitations were rather clear from the onset (I will elaborate upon this in the fourth section of this paper). Peschek (2009) develops a specific understanding of how secondary school-leaving examinations would have to change to increase their quality and comparability. When looking at examination usually proposed by individual teachers Peschek concedes that the examination has a strong focus on procedural knowledge, which is assessed in quite complex tasks. Helping enough students to pass these examinations more often then not means a quite direct “teaching to the tests”. The large variety of different, rather complex tasks individual examinations include makes it questionable whether students from one class would be able to pass the exam in another class. And, even more important, Peschek raises the question whether students have, at all, achieved any kind of “relational understanding” (Skemp, 1976) of the mathematical concepts and procedures addressed by these tasks. What educational policy makers and the mathematics education researchers involved in the pilot project and beyond could agree upon was the notion of *securing basic skills* (“Grundkompetenzen”) as the main measure to increase the quality and comparability of secondary school-leaving examinations. The project group from Klagenfurt was, however, not responsible for scaling-up “Zentralmatura” to the national level. There are quite a few further concessions and modifications that were made during the actual national implementation in 2014 and 2015.

## “Zentralmatura” and its specific conception of “mathematical literacy“

As I already stated, when revisiting the “Zentralmatura” pilot project and its conceptual framework, we have to concede that the project team was well aware that the aims stated in Roland Fischer’s conception of “höhere Allgemeinbildung” have a scope that is well beyond anything that can be assessed properly by means of a final written exam. But first let us clarify how this specific conception fits in with what is usually referred to as “mathematical literacy” at an international level.

Roland Fischer (2001) considers communication between experts and the lay public the greatest challenge general education faces in the modern democratic societies. Fischer argues that communication between experts and lay people is always asymmetrical: while expertise is precisely based upon the fact that the respective experts have a better understanding of the matter at hand than laypeople, it is mostly the laypeople who have to make decisions. For example: a surgeon usually has a better understanding of the benefits and risks of a surgery than the patient. Nonetheless, it is the patient who has to decide and give written notice of his “informed consent”. Likewise, politicians (as elected representatives of the public) may consult experts, but it is nonetheless their “job” to make and take responsibility for the actual decisions. Every democratic society is, after all, based upon the principle that in some way or form it is the (lay) public itself that decides upon its public matters. Nevertheless, that necessarily implies deciding upon proposals for problem resolution oneself would not be able to conduct and does not understand as well as the experts do. To Fischer, educating students to become well-informed laypeople should therefore focus on prospectively enabling them to make decisions about the importance of (mathematical) activities and problem resolutions even and especially in such cases in which they are not able to make (detailed) judgments about their technical correctness or to undertake the respective activities by themselves. For establishing a line between the professional study of a subject and its study for the purpose of Allgemeinbildung, Fischer (2001) distinguishes between three domains of knowledge in a subject:

"Firstly, the basic knowledge (notions, concepts, means of representation) and skills. Secondly, more or less creative ways of operating with knowledge and skills within applications (problem solving) or for the generation of new knowledge (research). Thirdly, reflection (What is the meaning/wherein lies the significance of these concepts and methods? What can be achieved with them, what are their limitations?)." (Fischer, 2001, p. 154)

Fischer then concludes that experts have to be well versed in all three domains, while the education of laypeople should focus on the first and third domain. One criticism towards Fischer's conception has been to dispute whether one can reflect meaningfully upon mathematics without actually "doing" the said mathematics. It is nonetheless the typical mode of confrontation with professional knowledge in a society that is based upon the division of labor. Furthermore, Fischer sees mathematical modelling and problem solving as important activities in the mathematics classroom. But, Fischer contests that being able to do (elaborate) mathematics can be a goal in itself for the purpose of general education. So, mathematical modelling or operating is seen by Fischer as a means to the end of acquiring *basic skills* as well as developing *reflective knowledge*, which is of particular interest for future citizens as well-informed lay public.

Considering different notions of "numeracy" and "mathematical literacy", we can link Roland Fischer's conception to the group of literacy conceptions Eva Jablonka (2003) describes as striving for either "mathematical literacy for evaluating mathematics" or even "mathematical literacy for social change". Jablonka (2003) differentiates such conceptions rather starkly from more pragmatic conceptions of "mathematical literacy" which e.g. the PISA framework uses and which Jablonka subsumes under the label "mathematical literacy for developing human capital". If we reconsider that introducing "Zentralmatura" can be framed as a measure of educational governance first and foremost, we would have to ask if more pragmatic conceptions of literacy focusing on building human capital, especially regarding the future workforce in STEM fields, would possibly have been a closer match.

## Revisiting the "Zentralmatura" pilot project

The "Zentralmatura" pilot project team was well aware that "Zentralmatura" could not fully implement everything Roland Fischer's conception called for. Fischer (2012) points out quite clearly that not both of these types of knowledge important to laypeople (basic skills and reflective knowledge) can be easily assessed under the constraints of a single written exam.

So the project group drew a distinction and they made a concession: "Zentralmatura" would have to mainly stick to the assessment of *basic skills*, which should be put front and centre within the examinations. Peschek et al. (2012, p. 82) mentions three different examination models the project group discussed in order to achieve such a central role of assessing basic skills. Model 1 would have restricted the whole examination to directly testing for the achievement of basic skills with items directly aimed at one specific basic skill (referred to as "tasks of type 1"). Such an examination would then result in a two-stage grading scale (pass vs. fail) for "Zentralmatura". Model 2 was conceived as a two-part examination: the first part would consist of "tasks of type 1" directly aiming at specific basic skills and the second part could at least try to test for the use of basic skills in more complex situations and context, where these skills were interconnected with each other and - to some extent - even the students' capacity to make value judgments about mathematics, for critique and for reflection could shine through. Model 3 was similar to Model 2, but only the first part of the examination would have been centralized, while the tasks for the second part would have been contributed by the individual teachers for their own classes. Within the pilot project, the educational policy makers only allowed for using Model 2, and the national implementation (see the section below) is also somewhat based on this model.

The second concession that was made concerns the identification of basic skills themselves, which is not solely based on their importance for promoting “mathematical literacy” or “Allgemeinbildung”, but also had to take the currently valid curriculum, intra-mathematical importance and social acceptability among teachers, parents and other “stakeholders” of mathematics education into account (Peschek et al. 2009, p.13).

Both Fischer (2012) and Peschek (2009) stated earlier on that they were absolutely aware that achieving meaningful “Allgemeinbildung” or “mathematical literacy” would still depend upon enough leeway for students and teachers to follow their interests to some degree, to explore mathematics and to engage in reflections about mathematics in meaningful ways that are just not possible within the confines of a standardized written exam. A rough guideline both Fischer (2012) and Peschek (2012) proposed was that within an “average” classroom securing basic skills necessary for passing the exams should not take more than half of the total teaching time. They also conceded that, especially during the first years of “Zentralmatura”, there would be a considerable number of classes which would have to dedicate more time to the training of those skills. Also, one would possibly have to evaluate whether the level of achievement “Zentralmatura” constitutes was appropriate, and what measures could be taken to support teachers and schools which fail to meet this level.

The outlook from The Pilot Project was cautiously optimistic, going by its end report (Peschek et al., 2012, p. 5-6): from a rather low and extremely heterogeneous level of performance considerable (and almost continuous) improvements were possible. Pilot tests gave teachers and students good orientation as to what basics skills were to be achieved. There were very satisfactory results in the first examination. 200 students from 11 classes took part and there were less than 10% failing grades. A problem that is mentioned in the report was that the production of good tasks for the second part was a major challenge for the project team – and would possibly be even harder in the future. The bottom line of the report was that a successful national implementation would depend on more than teachers adopting (or worse: just mimicking) examination modalities, i.e. by excessive use of multiple-choice questions. Mathematics teaching, according to Peschek et al. (2012), would have to change considerably in terms of both content and teaching methods.

## Observations from five years of national implementation of “Zentralmatura”

If we take a look at the transition from the pilot project to the national implementation of Zentralmatura, it did not get off to a good start. Its introduction at AHS was delayed for one year, with the widely professed reasoning that there was a lack of preparation (Schwarz, 2012), although each and every basic skill included in Zentralmatura was already a part of the curriculum for many years, so they should have been taught to students anyway. Some teachers and mathematicians came forward in stating their expressed lack of agreement with the aims of mathematical literacy and/or expressed concerns about a down levelling of more traditional mathematical demands (c. f. Brühl, 2012).

From the onset, part 2 of the actual Zentralmatura bore little resemblance to the pilot project. Although stated within overarching contexts, these problems intentionally consisted of isolated subtasks more or less directly aimed at “basic skills” again, just like in part 1, or unintentionally fell on the side of mere procedural knowledge/computational skills which the pilot project group declared less relevant for achieving “Allgemeinbildung” or “mathematical literacy”.

The percentage of failing grades has been highly volatile if one considers the numbers before the oral compensation exams (2015: 9.7%, 2016: 23.2 %, 2017: 11.8%, 2018: 22.5 %, 2019: 11.2 %). According to the well-informed sources from the Ministry of Education, this is the single most serious problem for the Ministry, because it counteracts the narrative of greater comparability and test fairness across different school years. Yet to this day, all the empirical educational research and field tests the Ministry has implemented to ensure the quality and fairness of “Zentralmatura” has not helped tackling this problem.

In 2018, there was a massive public outcry with a strong focus on (presupposed) influences of linguistic complexity and/or word count of the items. In public perception, Zentralmatura (AHS) is basically reduced to “tricky word puzzles” which both parents (Bundesverband der Elternvereine, 2018) and university STEM professors (Taschwer, 2018) criticise as unnecessary. This makes them somewhat strange bedfellows, because for most parents “Zentralmatura” is too hard on their kids, while for most STEM professors it demands too little of them.

## Central examinations and “mathematical literacy”: some cautious conclusions

I would like to close this paper by coming back to our initial question: What do we take away from these observations regarding the possibilities of sustainably implementing “mathematical literacy” through centralized secondary school-leaving examinations?

Looking at the pilot project, I would think one can say that if both teachers in the classroom are committed to the whole idea, and item development is overseen by well-versed experts in the field, considerable improvements in establishing a (basic) understanding and basic skills supporting mathematical literacy can be made in a relatively short period of time. But looking at the last 6 years of national examinations, I think one can also safely say that if teachers in the classroom are not committed to the whole idea and item development is less well overseen, centralized exams can only do so much. There is a number of reasons for this:

1. Teachers either do not have or at least do not see any leeway (Singer, 2015), but try to “drill and practice” basic skills in an unhelpful manner. Also, their actual leeway might be overstretched and/or used for activities unrelated/unhelpful to “mathematical literacy”.
2. Public and political support for “mathematical literacy” is just not there. In fact, “mathematics for all” is widely regarded more or less as a “necessary evil” to ensure a large enough recruitment base for the STEM-related workforce.
3. University STEM professors are a very vocal lobby group and – by and large – do not support “mathematical literacy”.
4. Actual test items sometimes do a less than stellar job in communicating the ideas behind the whole endeavour (Vohns, 2017).
5. There is simply no way of letting 20% or even more percent of students fail, so examinations will follow suit and not the other way around.

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# Sažetci / Abstracts

# Influence of Mathematics Content and Teaching of Mathematics on Students' Career Aspirations

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## Abstract

In Slovenia, the lack of interests of young people to become researchers or teachers of mathematics and other STEM (Science, Technology, Engineering, and Mathematics) subjects is noticeable, and that could be a real future threat for the Slovenian educational system. In this research, we observed the correlation between career aspirations and some constructs, i.e. the content of mathematical disciplines, mathematical contents from elementary and secondary school education, and the opinions about the teaching of elementary and secondary school. The research was made on the sample of 552 secondary school students from grades 3 and 4 of different gymnasiums and was part of the research already made by Šorgo and colleagues. In the survey, 15 different career streams were proposed and some of them include contents tightly connected with mathematics (e.g. education, engineering, finances, research, and development). Surprisingly, the results show that there is no correlation between the abovementioned constructs and any of the offered career streams.

***Keywords****: career aspirations, elementary school mathematics, mathematics teaching, secondary school mathematics, SEM analysis*

# Using Comparative Judgement to Assess Students' Mathematical Proficiency Regarding Quadratic Function

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## Abstract

Mathematical proficiency theory is a comprehensive view of successful mathematics learning. It includes five equally important and mutually interdependent strands that represent different aspects of a complex whole: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, Swafford & Findell, 2001). The procedural fluency strand, separately or in contrast with conceptual understanding, has been researched extensively. Researching aspects of strategy, metacognition and beliefs represent significant issues (Schoenfeld, 2007). Thus, some aspects of mathematical proficiency, e.g., conceptual understanding and adaptive reasoning, are neglected in most assessments. Generally, the strands of mathematical proficiency are investigated separately, contrary to the request to assess the aspects of mathematical proficiency as a whole, and not just its separate components (Burkhardt, 2007). This paper aims to address the deficiencies of standard mathematical testing and to enable assessment of more open and less structured tasks, by using an alternative approach to assessment, named comparative judgement. Comparative judgement offers the potential for assessing some global constructs such as mathematical ability and problem-solving (Jones & Inglis, 2015). The objective of this paper was to research the potential of such an assessment approach for investigating students' mathematical proficiency. To this end, mathematical proficiency regarding the quadratic function of one high school class was first tested and then assessed using the comparative judgement method.

***Keywords**: assessment; comparative judgement; high-school students; mathematical proficiency; quadratic function*

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# Mathematics Class as the Reference Point of Students' Attitudes and Beliefs in Mathematics Learning and Teaching

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## Abstract

In practice, attitudes and beliefs of students in mathematics teaching have been primarily observed as a consequence of students' success and family background. However, over the last thirty years, academic self-concept has been a topic of focus with recent discussions on how the perception of academic self-efficacy of students within the classroom is shaped. One of the most influential theories in this area is the “Big Fish in the Little Pond Effect” (BFLPE) (Marsh, 1984), which many researchers have confirmed worldwide on large and representative samples, thus opening many new issues and challenges that are particularly important for teaching mathematics. Within this field of research, the main focus is on how students' motivation and beliefs are affected by a social comparison among peers, and what impact they have on a variety of students’ mathematics-related beliefs, achievement values, task values, etc.

According to this theory, the average achievement of students negatively influences the academic self-concept of students, and thus, as the main challenge, it shows how to make more students achieve higher academic success. In other words, a student who is performing at an average level in higher-performing class may begin to find mathematics to be less useful, important, or interesting because he or she does not enjoy feeling such negative emotions (Wigfield & Eccles, 2000). Thus, for example, students from an objectively less demanding educational setting of some secondary vocational schools will often have more confidence and positive attitudes and beliefs in the context of some (or more) academic areas than their peers of equal capacities and knowledge attending objectively more demanding educational programs.

As the Croatian education system lacks research in the BFLPE theory for mathematics teaching, this paper aims to present the relevant literature, the theoretical background, and the specificity and importance of this phenomenon for mathematical education. The author describes how this phenomenon develops in different classroom environments of primary and secondary schools, and how its impact on mathematics teaching is heavier, and its consequences are on a larger scale than in other school subjects. In the end, the author states that contemporary math teaching should have pedagogical discussion and counseling as the focal point of supportive and the encouraging classroom atmosphere, which should be an integral part of teachers’ work to support students’ pursuit of educational aspirations and perspectives that are in line with their academic potentials.

***Keywords****: academic self-concept; mathematics class; pedagogical counselling; students' attitudes and beliefs*

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# Razredni odjel kao referentna točka stavova i uvjerenja učenika u nastavi matematike

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## Sažetak

Područje stavova i uvjerenja učenika u nastavi matematike se u praksi dominantno povezuje s uspjehom učenika i obiteljskim čimbenicima. Međutim, u zadnjih tridesetak godina pojačano se razmatra akademsko samopoimanje te kako se oblikuje percepcija akademske samoefikasnosti učenika u okvirima razrednog odjela. Među najutjecajnijim teorijama u tom području je Big Fish in Little Pond Effect (BFLPE) (Marsh, 1984) koju su brojni istraživači diljem svijeta potvrdili na velikim i reprezentativnim uzorcima te time otvorili mnoga nova pitanja i izazove koji se pokazuju naročito značajnima za nastavu matematike. U tom je području istraživanja glavni cilj odrediti na koji način motivacija i uvjerenja učenika ovise o socijalnim usporedbama vršnjaka te kakav je njihov utjecaj na svekolika uvjerenja i vrijednosni sustav učenika u kontekstu nastave matematike.

Prema toj teoriji prosječno postignuće učenika negativno utječe na njegovo akademsko samopoimanje pa je temeljni izazov kako omogućiti većem broju učenika da se ostvare u odgojno-obrazovnom sustavu. Drugim riječima, učenik prosječnoga školskog uspjeha među vršnjacima u razrednom odjelu iznadprosječnog uspjeha može početi smatrati matematiku manje važnom, manje korisnom ili manje zanimljivom jer se njome suočava s nizom negativnih emocija s kojima se teško nosi (Wigfield i Eccles, 2000). Tako će primjerice učenici iz nekog objektivno manje zahtjevnogaa odgojno-obrazovnog usmjerenja neke srednje strukovne škole često imati više samopouzdanja te pozitivnih stavova i uvjerenja u kontekstu nekog akademskog područja, ili više njih, negoli njihovi vršnjaci podjednakog kapaciteta i znanja koji pohađaju objektivno zahtjevnije odgojno-obrazovne programe.

Kako u hrvatskom odgojno-obrazovnom sustavu nedostaje istraživanja BFLPE teorije za nastavu matematike, cilj ovog rada jest prikazati relevantnu literaturu, teorijsku utemeljenost te specifičnost i važnost ovog fenomena za matematičko obrazovanje. Autor opisuje kako se u različitim razrednim ozračjima osnovnih i srednjih škola ovaj fenomen razvija te kako u nastavi matematike on ima utjecaje širih okvira i posljedice većih razmjera nego u drugim predmetnim područjima. Zaključno ističe kako bi suvremena nastava matematike u središtu svojih namjera trebala imati pedagoški razgovor i savjetovanje kao okosnicu poticajnoga i ohrabrujućega razrednog ozračja koje je nužno kako učenicima ne bi bile uskraćene obrazovne aspiracije i perspektive usklađene s njihovim potencijalima.

**Ključne riječi:** *akademsko samopoimanje; nastava matematike; pedagoško savjetovanje; stavovi i uvjerenja učenika*

# Systematic Review as a Research Method: A Case of Professional Development of Mathematics Teachers

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## Abstract

This study utilized a systematic review of literature as the main research method. The systematic review consists of precisely defined steps to ensure research rigor. First, we formulated appropriate research questions. Second, we defined the search terms and selected databases. Third, we used inclusion and exclusion criteria, which guided us in the further literature search. Fourth, we evaluated the scientific quality of the obtained publications using predefined quality criteria. Only studies that met the quality requirements were included in this review. Finally, data answering the research questions were extracted. Our aim was to identify studies that examined the professional development of mathematics teachers with an influence on student achievements. In this process, we identified 22 studies connected with our research question. In the reviewed studies, we examined whether professional development influenced student achievements, the effect size of achievement, and we classified the models used in professional development according to the autonomy given to the teacher in the professional development. Most professional development models were transmissive, some combined features of malleable and transmissive models, while characteristics of only malleable models were present in few studies. Half of the studies described professional development that did not impact student achievements. Our review showed that many studies had small effect size on student achievements in comparison with the effect size of 0.40, Hattie's "hinge point" for educational interventions.

***Keywords:*** *effect size; mathematics teachers; professional development; systematic review*

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# Sustavni pregled literature kao istraživačka metoda: primjer stručnog usavršavanja učitelja matematike

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## Sažetak

U ovoj se studiji koristio sustavni pregled literature kao glavna istraživačka metoda. Takav pregled literature sastoji se od točno propisanih koraka kako bi se osigurala strogost istraživanja. Prvo je potrebno formulirati odgovarajuća istraživačka pitanja. Drugo, potrebno je definirati pojmove za pretraživanje i odabrati baze podataka. Treće, u daljnjem pretraživanju literature potrebno je rabiti kriterije za uključivanje i isključivanje publikacija. Potrebno je ocijeniti i znanstvenu kvalitetu publikacija pomoću unaprijed definiranih kriterija kvalitete. U ovaj su pregled uključene samo one studije koje su zadovoljile kriterije kvalitete. Naposljetku su izdvojeni podatci koji odgovaraju na postavljena istraživačka pitanja. Cilj ovog istraživanja bio je identificirati one studije koje su istraživale stručno usavršavanje i multikomponentne intervencije učitelja matematike s utjecajem na postignuća učenika. U ovom procesu identificirane su 22 studije povezane s istraživačkim pitanjem. U pregledanim publikacijama ispitalo se kako je stručno usavršavanje utjecalo na postignuće učenika i provjerilo se veličinu učinka učeničkih postignuća. Dodatno, klasificirani su modeli stručnog usavršavanja opisani u tim studijama s obzirom na razinu autonomije koja je dana učiteljima tijekom usavršavanja. Većina usavršavanja bila su transmisijskog tipa, manji broj usavršavanja imao je kombinirane značajke prilagodljivog (*malleable*) i transmisijskog modela, dok je mali broj studija imao karakteristike samo prilagodljivog modela. Zanimljiv je podatak da polovina pregledanih studija opisuje stručno usavršavanje koje nije utjecalo na postignuća učenika. Također su mnoge studije imale malu veličinu učinka na postignuća učenika, u usporedbi s veličinom učinka 0,40 koju Hattie smatra prijelomnom točkom u obrazovnim intervencijama.

***Ključne riječi****: postignuća učenika; stručno usavršavanje; sustavni pregled; učitelji matematike*

# The Influence of Students' Personal Characteristics on Their Mathematical Homework Performance

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## Abstract

A number of factors, including involved persons, can influence student's homework performance. Factors are connected to teachers who set up homework tasks, and sometimes also to students' parents or carers who participate in homework completion, but in any case, many factors originate from students' characteristics. In the presented study, we focus on the latter. We are studying if and how students' self-esteem, students' mathematical confidence, and students' attitudes toward learning mathematics influence homework performance. We focus on following students' homework behaviours: the effort that a student invests in homework, the share of homework completed by a student, and students' optimization of time while doing homework. As several studies suggest, all the abovementioned students' homework behaviours could be in positive correlation with students' mathematics achievements (e.g. Núñez et al., 2015; Trautwein, & Lüdtke, 2007). The paper presents the results of an international survey involving 729 students from the final three grades of elementary education in Slovenia and Croatia. The results show that the self-esteem of both Slovenian and Croatian students, measured with the Rosenberg scale (Rosenberg, 1965), is not correlated with mathematics homework completion, homework time optimization, or even with students' effort invested in homework. The students' mathematical confidence or their perceptions of success in mathematics, and the students' attitudes toward learning mathematics were measured by expressing the level of students' acceptance, with the statements used in the TIMSS 2015 study (IEA, 2013). For Slovenian students, the results show a very weak negative relationship between students' mathematical confidence and homework completion, students' effort and time optimization while doing homework. For the students from Croatia, these relationships cannot be confirmed. However, for the students from both countries, the attitudes toward learning mathematics are in a weak, or respectively, in a moderately positive correlation with all of the mentioned students' homework behaviours that influence mathematical achievements. These results imply that the promotion of students' positive attitudes towards learning mathematics should be more emphasised than building up students' mathematical confidence.

***Keywords:*** *attitudes towards learning mathematics; homework; mathematical confidence; mathematics; self-esteem.*

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# An Analysis of 6th Graders' Abilities to Relate Contextual and Non-Contextual Problems

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## Abstract

Mathematical problems could be contextual, relating to everyday real life in comparison to non-contextual, where mathematical concepts prevail and solutions to such problems have a more intellectual than practical value for the solver. We usually relate contextual problems to mathematical literacy that is defined as the ability of an individual to recognize mathematics in everyday life situations and can use mathematical knowledge to satisfy his/her own needs. This means that a student can solve problems in different contexts such as personal, social, professional, and scientific that are connected to mathematics in a specific way. The main focus of our research was to find out how 6th-grade students recognize mathematical concepts in problems based on everyday life contexts, so-called contextual problems. We were interested in analysing the strategies used by 6th graders when faced with non-contextual and contextual problems. They were confronted with six problems, 3 pairs of non-contextual and contextual ones (the mathematical ideas were: common multiples, the power, and parts and wholes). The mathematical concept in a non-contextual problem was visible, whereas, in the contextual problem, the same mathematical idea was not directly visible; the solver had to recognise it within the context. The results have shown that the connection between success in both types of problems is not obvious, which means that success in non-contextual problems is not necessarily a prerequisite for success in the contextual ones. We found many different solutions that clearly show that a non-contextual problem in the pair of problems was solved better than the contextual one and vice versa. We also observed that some pupils were able to use the same strategy in both problems in the pair, but some students approached them differently. The results have also shown that students have problems with the heuristic 'working backward', and therefore, the pair of problems demanding that particular strategy was the most poorly solved. Generally, it was confirmed that being proficient in basic mathematical skills and having basic knowledge of mathematical concepts are two conditions for mathematical literacy, but on the other hand, they are not necessarily sufficient conditions for making a transfer to a contextual everyday life problem. Mathematical literacy is a specific part of mathematics, consisting of the use of mathematical knowledge in contexts that are not structured in the same way as the usual school contexts, and therefore, some specific attention is needed in school to improve solving such problems. We propose teaching heuristics as a good 'start'.

***Keywords**: 6th grade students; contextual problem; mathematical literacy; mathematical problem; strategy*

# Study of the Developmental Course Knowledge of Mathematical Competence of Children in Preschool Teachers and Designing the Educational Context for the Development of Mathematical Competence

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## Abstract

The aim of this study was to determine the level of knowledge of the developmental course of mathematical competence of children in preschool teachers, and to examine the correlation between the knowledge of the developmental course of mathematical competence of children in preschool teachers and some aspects of the design of the educational context for the development of the mentioned competence in children.

The study involved 66 participants, that is, preschool teachers. Participation in this survey was anonymous and voluntary. Participants filled in the Questionnaire for Preschool Teachers consisting of three parts: the introductory part of the Questionnaire that contains questions about the socio-demographic characteristics of the participants, followed by the Questionnaire about knowledge of the development of mathematical competence and Questionnaire about the educational context for the development of mathematical competence.

The basic statistical parameters of preschool teachers' responses to the Questionnaire about knowledge of the development of mathematical competence (QMC) and the Questionnaire about the educational context for the development of mathematical competence (QE-U) were established. There is a higher percentage of accurate QMC responses on the issues related to the development of mathematical competence of children in the area of numeracy and numerical symbols for numbers than the issues related to the knowledge of ordinal numbers, addition/subtraction, and division of sets. Participants' responses to the questions about educational context for the development of mathematical competence were calculated: 53.03 percent of participants stated that they had not carried out a project in the area of early development of mathematical competence of children, and 80.30 percent stated that they had not participated in a professional assembly/education about the educational context for the development of mathematical competence in the pedagogical year 2017/2018.

No statistically significant correlation has been established between the overall score on the Questionnaire about knowledge of the development of mathematical competence and the single question of Questionnaire about the educational context for the development of mathematical competence.

The participants were divided into two subgroups according to the working age: one subgroup was composed of the participants who had up to 5 years of experience working as preschool teachers, while the other subgroup consisted of the participants with 6+ years of work experience. The obtained t-test indicates that the difference between the participants of different lengths of work experience in relation to the overall score on QMC is statistically significant. Participants with up to 5 years of work experience achieved a higher score on this Questionnaire.

In future research, it would be useful to investigate on a larger sample the correlation between other variables relevant for the development of mathematical competence by children (e.g., preschool teachers' beliefs, educational methods, etc.).

***Keywords:*** *educational context; mathematical competence by children; preschool teachers*

# Ispitivanje odgojiteljskog poznavanja razvojnog tijeka matematičke kompetencije djece predškolske dobi i stvaranja odgojno-obrazovnih uvjeta za njezin razvoj

Goranka Matić Šarić
*Dječji vrtić Dugo Selo*

## Sažetak

Cilj ovog istraživanja bio je utvrditi stupanj poznavanja razvojnog tijeka matematičke kompetencije djece predškolske dobi kod odgojitelja te ispitati postoji li povezanost između poznavanja razvojnog tijeka matematičke kompetencije djece s nekim aspektima oblikovanja odgojno-obrazovnog konteksta za razvoj navedene kompetencije kod djece.

U istraživanju je sudjelovalo 66 sudionika, odnosno, odgojitelja. Sudjelovanje u ovom anketnom istraživanju bilo je anonimno i dobrovoljno. Sudionici su ispunili *Upitnik za odgojitelje* koji se sastojao od tri dijela: uvodni dio s pitanjima o socio-demografskim obilježjima sudionika, dio poznavanja razvoja matematičke kompetencije djece te dio o odgojno-obrazovnim uvjetima za razvoj matematičke kompetencije.

Utvrđeni su osnovni statistički parametri odgovora sudionika na *Upitniku poznavanja razvoja matematičke kompetencije djece* (*UMK*) i na *Upitniku o odgojno-obrazovnim uvjetima za razvoj matematičke kompetencije djece* (*UO-O*). Veći je postotak točnih odgovora na UMK-u za pitanja koja se odnose na razvoj matematičke kompetencije djece na području usvajanja brojenja, naziva za brojeve i pisane simbole za brojeve nego za pitanja koja se odnose na poznavanja rednih brojeva, zbrajanje/oduzimanje i podjelu skupova. Izračunati su postotci odgovora sudionika na pitanja o stvaranju odgojno-obrazovnih uvjeta za razvoj matematičke kompetencije: 53,03 % sudionika navodi da nisu proveli projekt iz područja ranog razvoja matematičke kompetencije djece, a njih 80,3 % navodi da nisu sudjelovali na stručnom skupu/edukaciji o stvaranju uvjeta za razvoj matematičke kompetencije djece u pedagoškoj godini 2017./2018.

Nije utvrđena statistički značajna povezanost između ukupnog rezultata na *Upitniku poznavanja razvojnog tijeka matematičke kompetencije djece* i odgovora na pojedino pitanje na *Upitniku o odgojno-obrazovnom uvjetima za razvoj matematičke kompetencije*.

Sudionici su prema godinama radnog staža podijeljeni u dvije podskupine: jednu su podskupinu činili sudionici koji imaju do 5 godina radnog staža na poslovima odgojitelja, a drugu oni sa 6 i više godina radnog staža. Dobivena vrijednost t-testa ukazuje da je razlika između sudionika različite dužine radnog staža na poslovima odgojitelja prema ukupnom rezultatu na UMK-u statistički značajna. Sudionici koji imaju do 5 godina radnog staža postižu viši rezultat na tom upitniku.

U budućim bi istraživanjima bilo korisno na većem uzorku sudionika istražiti povezanost drugih relevantnih varijabli (uvjerenja odgojitelja; odgojno-obrazovne metode i drugo) za poticanje razvoja matematičke kompetencije djece.

***Ključne riječi****: matematičke kompetencije djece; odgojitelji; odgojno-obrazovni kontekst*

# Influence of Textbooks on 3rd-Grade Students' Achievements on the SPUR Test

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## Abstract

A textbook is a basic didactic material in the teaching of mathematics, and as such, has a great influence on students' development. In this study, we analyzed mathematics textbooks for the 3rd grade of elementary school, in the field of fragments to determine the representation of tasks according to SPUR. Next, we determined the impact of textbook sets on students' achievements on the SPUR test and examined the extent to which teachers use textbook sets in mathematics classes. The SPUR test is a multi-dimensional approach to learning mathematics that encourages the development of students through four dimensions: Skills, Properties, Uses, and Representations. The skills include knowledge of procedures for solving a task, the properties include mathematical principles and facts, the uses include knowledge in everyday life, and the representation includes the use of visual representations (Thompson & Kaur, 2011).

The sample consists of textbook sets of mathematics for the 3rd grade of primary school from three publishing houses that are mostly used on the territory of the Sombor School Administration, 125 students of the third grade and their teachers from the territory of the Sombor School Administration. The instruments are a checklist, a test for students, and a teacher survey created for this research.

The analyses show that the textbook sets differ significantly in terms of the number of tasks by SPUR. The textbook sets mostly contain tasks related to skills, and in the least, tasks related to properties. The results on the SPUR test show that students achieved the best results on the tasks related to skills, and the worst results on the tasks related to the use. Also, the results from the student achievement test show that the students who use a textbook that contains the fewest tasks by SPUR have the best results, which leads us to the conclusion that the number of tasks does not guarantee good results on the test. The survey results show that teachers are satisfied with the number and quality of the tasks, that they have done more than 75 percent of the tasks from the textbooks with the students, and that they give homework assignments from the workbook, as well as the tasks they prepare on their own. In addition to textbooks, other factors, such as teachers' competences, have a major impact on students' achievements.

***Keywords***: *SPUR; students' achievements; textbook; use of textbooks; workbook*

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# Utjecaj udžbenika na postignuća učenika 3. razreda na SPUR testu

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## Sažetak

Udzbenički komplet osnovni je didaktički materijal u nastavi matematike i kao takav ima velik utjecaj na razoj učenika. SPUR je multidimenzionalni pristup učenju matematike koji potiče razvoj učenika u četirima dimenzijama: vještine (eng. *skills*), osobine (eng. *properties*), upotreba (eng. *uses*) i predstavljanje (eng. *representations*). Vještine podrazumijevaju poznavanje procedura za rješavanje zadataka, osobine podrazumijevaju matematičke principe i činjenice, upotreba podrazumijeva primjenu znanja u svakodnevnom životu, a predstavljanje upotrebu vizualnih prikaza (Thompson i Kaur, 2011). U ovom radu analizirani su udzbenički kompleti matematike za 3. razred osnovne škole iz oblasti Razlomci s ciljem da se utvrdi zastupljenost zadataka po SPUR-u, zatim je ispitan utjecaj udzbeničkih kompleta na postignuća učenika na SPUR testu te u kojoj se mjeri učitelji koriste udzbeničkim kompletima na satu matematike.
Analizirani su udžbenički kompleti matematike za 3. razred osnovne škole, koji se najviše koriste u Školskoj upravi Sombor. Uzorak ispitanika činilo je 125 učenika 3. razreda i njihovi učitelji iz Školske uprve Sombor. Ispitivanje je provedeno pomoću upitnika za učenike i ankete za učitelje, sastavljenih za potrebe ovog istraživanja.
Analize pokazuju da se udžbenički kompleti medjusobno znatno razlikuju po broju zadataka po SPUR-u. Udžbenički kompleti najviše sadrže zadatke koji se odnose na vještine, a najmanje zadatke koji se odnose na osobine. Rezultati SPUR testa pokazuju da su učenici najuspješniji u zadatcima koji se odnose na vještine, a najmanje uspješni u zadatcima koji se odnose na upotrebu. Također, rezultati testa postignuća učenika pokazuju da učenici koji se koriste udžbeničkim kompeletom s najmanje zadataka po SPUR-u imaju najbolje rezulate, što navodi na zaključak da broj zadataka ne jamči i dobre rezultate na testu. Rezultati ankete pokazuju da su učitelji zadovoljni brojem i kvalitetom zadataka, da su s učenicima ostvarili više od 75 % zadataka iz udžbeničkih kompleta i da se dodatno koriste zadatcima iz radne biljeznice i zadatcima koje sami pripremaju. Pored udžbeničkih kompleta, velik utjecaj na postignuća učenika imaju i drugi faktori, poput kompetencija učitelja.

***Ključne riječi****: postignuća učenika; radna bilježnica; SPUR, udžbenik; upotreba udžbenika*

# Metacognitive Feelings and the Illusion of Linearity

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## Abstract

The illusion of linearity is the tendency to comprehend certain sizes as linearly related, even when such understanding is not justified. This illusion is manifested in providing linear answers in non-linear tasks. It occurs due to linearity heuristic, that is, intuitive and automatic, but sometimes an erroneous and biased type of information processing. It is assumed that mathematical education that is focused on procedural knowledge and routine expertise is important in the appearance and maintenance of the illusion of linearity. In order to override linearity heuristic, people need to scrutinize information. The trigger for engaging this analytical and rational type of information processing are metacognitive feelings that represent experiential experiences informing a person about cognitive processing, and serve as the interface between some task and the person. For instance, when people have a lower metacognitive feeling of rightness (FOR), they are more inclined to deeply analyse their responses. The role of metacognitive feeling, as well as conceptual and procedural knowledge, in the occurrence of the illusion of linearity, has not been thoroughly explored. Therefore, our research aimed to examine whether interventions focused on students' conceptual and procedural knowledge would affect their FOR, and consequently, the illusion of linearity. The participants were high-school students (N=908) who solved five linear and five non-linear tasks randomly presented on the computer and were given different instructions about task solving. While some students had conceptual instruction, the other had procedural instruction, and both instructions were grounded in the productive failure method. There was also a control group. In our study, students answered in accordance with the illusion of linearity. The results showed that conceptual and procedural instructions decreased the illusion of linearity. Students who had lower FOR were more inclined to analyse their answers and spent more time on thinking about tasks. It can be concluded that metacognitive feelings have an important role in understanding and decreasing of the illusion of linearity.

***Keywords****: conceptual knowledge; procedural knowledge; metacognitive feelings; the illusion of linearity*

# Primary Education Pre-Service Teachers' Achievement Goals in Mathematics and Their Approach to Learning and Teaching Mathematics: A Person-Centered Analysis

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## Abstract

In Croatia, primary education teachers are trained as generalists, and mathematics is only one of several different subjects that they teach. That means that, when they chose their future profession, they were not necessarily drawn by their interest in becoming mathematics teachers. On the other hand, it is very important that they have good mathematics teaching skills, along with positive attitudes toward mathematics, and are motivated to teach it to their students. Therefore, we were interested in finding out whether there was an "at-risk group" of future primary education teachers that began their studies with low motivation for learning mathematics, and whether they would have different attitudes toward learning and teaching mathematics in comparison to more motivated groups. The participants were 325 primary education students. In their first year of studies, we collected data on achievement goals in mathematics that they held in high school, motivation for learning mathematics during their studies, mathematical epistemic beliefs, mathematics anxiety, and preferences for different types of mathematical problems. We also assessed their mathematics performance. In their third year of studies, we collected data on their mathematical epistemic beliefs, mathematics anxiety, and mathematics teaching efficacy beliefs. The results of the cluster analysis showed that we could put primary education pre-service teachers in three groups according to the profiles of their achievement goals in high school: (1) dominant mastery goals, (2) all goals high, (3) all goals low. We identified the third group as the "at-risk group". The results of the ANCOVA, with achievement goals profile as an independent variable, showed that different groups differ in the motivation for learning mathematics during their studies, preferences for different types of mathematical problems, epistemic beliefs, mathematics anxiety, and teaching efficacy beliefs, even when controlling for their mathematics performance. The "at-risk group" had the least adaptive beliefs. However, the differences between the groups were not large and tended to be less prominent in the third year of studies than in the first. Hence, it would be interesting to further explore the role of initial teacher education in forming adaptive beliefs in mathematics.

***Keywords****: initial teacher education; mathematics; achievement goals; motivation; teacher beliefs*

# Theme-Based Approach to Mathematics Teaching in Lower Primary School

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## Abstract

At the very beginning of schooling, students tend to create either positive or negative attitudes toward mathematics. The main reason for the emergence of negative attitudes has not yet been established, but it is becoming evident that mathematics is a difficult subject for students who are unable to achieve success and master it because of how the content is presented (Arambašić, Vlahović-Štetić, & Severinac, 2005). Different forms of teaching that have some important common features are more and more described in professional literature, and used in the modern teaching process. A student who is actively involved in creating his/her knowledge, is interested, motivated by curiosity, and his/her knowledge is based on understanding. Increasing efforts are made to bring the teaching process to the child's needs, moving away from traditional teaching. One of the main teaching strategies that follow children's needs, and the interpretation of the world as a whole, is integrated teaching. This implies planning and organizing the teaching process in which we connect different educational areas to achieve a deep and comprehensive understanding of certain content. Integrated teaching through different forms achieves the educational tasks of modern teaching. Theme-based teaching, as a form of integrated teaching, puts a certain topic in the center around which we build activities in an interdisciplinary way, and form a Math lesson known as a themed lesson.

Within this paper, the theme-based Math lessons in grades 1, 2, 3, and 4 were presented to show and analyze whether math could be brought closer to the students through the theme-based approach. The teaching process in each class was shaped to a particular subject that was close to the students and adapted to their age, to enable the implementation of natural theme-based teaching. The analysis of evaluation and self-evaluation sheets showed that the students were generally motivated for Math lessons, and the teachers were motivated for using the theme-based approach.

***Keywords:*** *integrated teaching; mathematics; Math class; theme-based class; theme Math lessons*

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# Tematski sat matematike u razrednoj nastavi

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## Sažetak

Upravo na samom početku školovanja učenici stvaraju pozitivne ili negativne stavove prema matematici. Glavni razlog nastanka negativnih stavova još nije utvrđen, ali s vremenom sve više jača uvjerenje kako je matematika učenicima težak predmet u kojem ne mogu biti uspješni i savladati ga zbog načina na koji im se prezentira nastavni sadržaj (Arambašić, Vlahović-Štefić i Severinac, 2005). U stručnoj literaturi sve se više opisuju, a u suvremenom nastavnom procesu i koriste, različiti oblici nastave koji imaju neka bitna zajednička svojstva. Učenik koji aktivno sudjeluje u stvaranju vlastitih spoznaja, zainteresiran je i motiviran znatiželjom, a znanje mu je utemeljeno na razumijevanju. Sve je više nastojanja da se nastavni proces ostvaruje prema djetetovim potrebama, udaljavajući se od tradicionalnog poučavanja. Integrirano poučavanje jedan od oblika poučavanja usmjerenih na dijete i njegove potrebe. Ono podrazumijeva planiranje i organiziranje poučavanja u kojem se povezuju različita obrazovna i odgojna područja s ciljem postizanja dubokog i cjelovitog razumijevanja određenog sadržaja. Integrirano poučavanje različitim načinima ostvaruje odgojno-obrazovne zadaće suvremene nastave. Tematsko poučavanje, kao jedan od oblika integriranog poučavanja, u središte postavlja određenu temu oko koje se interdisciplinarno grade nastavne aktivnosti i oblikuje nastavni sat, u nastavi matematike poznatiji kao tematski sat.

U sklopu ovog rada provedeni su tematski sati matematike u 1., 2., 3. i 4. razredu s ciljem utvrđivanja može li se učenicima nastava matematike dodatno približiti tematskim poučavanjem. Nastavni proces u svakom razredu oblikovan je prema određenoj temi koja je bliska učenicima i prilagođena njihovoj dobi, kako bi se omogućila provedba prirodnog tematskog poučavanja. Analiza evaluacijskih i samoevaluacijskih listića pokazala je kako su učenici cjelokupno bili motivirani za nastavne sate, kao i učiteljice za korištenje tematskog sata matematike u svojoj nastavi.


***Ključne riječi:*** *integrirano poučavanje; matematika, nastava matematike; tematsko poučavanje; tematski sat matematike*

# Teaching of Initial Multiplication Concepts and Skills in Croatian Textbooks

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## Abstract

Since teaching is a complex system, rooted in a certain cultural script (Stiegler & Hiebert, 1999), in describing it, it is necessary to step out of this cultural frame. Only then one can notice some of its attributes that appear to be self-evident from the inside.

For this reason, we compared the teaching of initial multiplication concepts and skills, up to the multiplications table, in two series of textbooks from Croatia and Singapore. In the analysis of the textbooks, we used an adapted framework from Charalambous, Delaney, Hsu, and Mesa (2010) that looks at a textbook as an environment for the construction of knowledge of a single mathematical concept.

Preliminary findings (Baković, Trupčević, & Valent, 2019) indicate that in Croatia, learning of initial multiplication concepts and skills heavily relies on practice, without given support in underlying constructs and representations or different multiplication strategies. Hence it is expected that the multiplications table is understood by Croatian students as something that is expected to be memorized, and not something to be understood and developed on their own.

***Keywords****: concept construction; textbook analysis; multiplication*

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# Poučavanje početnih koncepata i vještina vezanih uz množenje u hrvatskim udžbenicima

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## Sažetak

Kako je poučavanje kompleksan sustav utemeljen na određenom kulturnom predlošku, potrebno je iskoračiti iz danoga kulturnog okvira da bi ga se opisalo. Tek tada postaju uočljivi neki njegovi atributi koji se iznutra čine očiglednima.
Zbog toga smo usporedili poučavanje početnih koncepata i vještina vezanih uz množenje, uključujući i tablicu množenja, u dvjema serijama udžbenika iz Hrvatske i Singapura. U analizi udžbenika koristili smo prilagođeni okvir autora Charalambous, Delaney, Hsu i Mesa (2010) koji na udžbenik gleda kao na okruženje za konstrukciju znanja o pojedinom matematičkom konceptu.
Preliminarni rezultati istraživanja (Baković, Trupčević i Valent, 2019) pokazuju da se u Hrvatskoj učenje početnih koncepata i vještina vezanih uz množenje u velikoj mjeri oslanja na uvježbavanje, bez podrške u vidu različitih konstrukata i reprezentacija vezanih uz množenje ili različitih strategija računanja pri množenju. Stoga je za očekivati da množenje i tablicu množenja hrvatski učenici shvaćaju kao nešto što treba naučiti napamet, a ne kao nešto što treba razumjeti i do čega treba moći samostalno doći.

***Ključne riječi**: analiza udžbenika; množenje; razvoj koncepata*

# Unseen Connections: Problem Graphs in Demonstrating and Improving Teachers' Design

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## Abstract

Teachers' design work is arguably a decisive factor in the success of the instructional process, and thus, intensely targeted by research. Recently, Gueudet, Pepin, & Trouche (2017) proposed ten questions for a better understanding of teachers' designer activity. Nevertheless, we find that certain aspects of this activity, especially long-term process planning, is considered difficult by teachers and hardly accessible by researchers. It is debated to what extent it is the duty of the teacher at all – some argue that this responsibility rather belongs to curriculum and textbook developers. Our ongoing research suggests that in some cases, a vast amount of content knowledge for teaching and horizon content knowledge (Ball et al., 2008) is embedded into the instructional sequences, but it is difficult to discern and even more, to convey this knowledge to fellow teachers, let alone outsiders. To boost this process and to help teachers' design work, the problem graph, a representational and design tool is being developed in the frames of this research project. This research is indigenous in the cultural context of the Hungarian mathematics education tradition that tends to be more problem-based than most (Gosztonyi, 2018; Gosztonyi et al. 2018), but these results are of relevance to the international community.
In my contribution, I will present an extensive and complex example for problem graphs and some results of the pilot experiments with teachers working with problem graphs.

***Keywords***: *design capacity; horizon content knowledge; learning trajectories; mathematical knowledge for teaching; problem graph*

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